EXISTENCE AND APPROXIMATION OF SOLUTIONS OF DIFFERENTIAL EQUATIONS

By

Imran Talib

UNIVERSITY OF MANAGEMENT AND TECHNOLOGY

SCHOOL OF SCIENCE, DEPARTMENT OF MATHEMATICS

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External Examiner:

Research Supervisor:

Assistant Professor
Dr. Naseer Ahmad Asif

Examing Committee:

UNIVERSITY OF MANAGEMENT AND TECHNOLOGY

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Dedicated to

Hazrat Wasif Ali Wasif, **Sir Muhammad Afzal Hussain**,

Loving Parents

and

Dear Wife

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Abstract

The nonlinear coupled boundary value problems (BVPs) have very important and interesting aspects in the kingdom of Nonlinear Analysis due to not only the theoretical aspects but also the applications which they have in almost every field of science. Problems with coupled boundary conditions (BCs) appear while studying mathematical biology, Sturm-Liouville problems, reaction diffusion phenomena, chemical systems, and Lokta-Volterra models.

This thesis has two parts. In the first part, the existence results are established for the first–order and the second–order nonlinear coupled BVPs subject to nonlinear coupled BCs. Also in the same part, the existence results are established for the second–order nonlinear coupled BVPs when the nonlinear functions have dependence on the first-order derivative.

Multiple approaches are available in the literature to investigate the existence of solutions of nonlinear BVPs, but lower and upper solutions (LUSs) approach is one of the strongest. In this approach the original problem is modified logically to a new problem, known as modified problem, then the theory of differential inequalities with the combination of well-known existence results are applied to establish the existence of solution of the modified problem. Finally the solution of the modified problem leads to the solution of the original problem. Moreover in the first part of the thesis the treatment of the many different first-order and the second-order nonlinear BVPs are unified by developing the idea of coupled LUSs. Under this idea, some monotonicity assumptions are imposed on the arguments of the nonlinear BCs in the presence of the existence of a lower solution and an upper solution to unify the classical existence results for very important types of BVPs, like periodic, anti-periodic, Dirichlet, and Neumann. Several examples are discussed to support the theoretical results.

The subject fractional calculus being a generalization of integer-order calculus has numerous applications in almost every field of science. Due to the intensive use of fractional order differential problems (FODPs) in almost every field of science based including, but not limited to, fluid dynamics, physics, aerodynamics, chemistry, mathematical biology, image processing, and psychology, there is a strong motivation for the researchers to develop the reliable and efficient numerical methods to find the approximate solutions of FODPs.

In the second part of the thesis, we consider a generalized class of multi–terms fractional order partial differential equations (FOPDEs) and their coupled systems. We develop a new numerical method and generalize the corresponding Jacobi operational matrices of integrals and derivatives considered on a rectangular plane. By means of the operational matrices, the considered problem of fractional order is reduced to an algebraic one. Being easily solvable, the associated algebraic system leads to finding the solution of the considered problem of fractional order. Validity of the method is established by comparing our Matlab software simulations based obtained results with the exact solutions in the literature yielding negligible errors.

List of Publication From Thesis

Published papers

[1] I. Talib, N.A. Asif, C. Tunc, *Existence of solutions to second-order nonlinear coupled systems with nonlinear coupled boundary conditions*, Electron. J. Differential Equations **2015**(2015), no. 313, 1–11.

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[3]I. Talib, N.A. Asif, C. Tunc, *Coupled lower and upper solution approach for the existence of solutions of nonlinear coupled system with nonlinear coupled boundary Conditions*, Proyecciones Journal of Mathematics **35**(2016), 99–117.

[4] N.A. Asif, I. Talib, *Existence of solutions to a second order coupled system with nonlinear coupled boundary conditions*, American Journal of Applied Mathematics, **3**(3-1)(2015), 54–59.

List of Submitted Papers From Thesis

Submitted papers

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[2] I. Talib, H. Khalil, N.A. Asif, C. Tunc, F.B.M. Belgacem, *On Multi-term fractional partial differential equations mixed type coupled system approximate solutions*, Found. Comput. Math.

[3] I. Talib, F.B.M. Belgacem, N.A. Asif, H. Khalil, *On mixed derivative types high dimensional multi-term fractional partial differential equations approximate solutions*, ICNPAA 2016, University of La Rochelle, France.

Preface

This thesis comprises of two parts and addresses the different types of problems having some very interesting and productive impact in the world of nonlinear coupled boundary value problems (CBVPs).

The first part deals with the existence criterion for the solutions of nonlinear coupled systems of ordinary differential equations (ODEs) together with nonlinear coupled boundary conditions (CBCs). In the kingdom of differential equations, the study of nonlinear CBVPs have very important and interesting aspects for the researchers due to not only the theoretical aspects but also the rich applications in the existing literature. The applications of CBCs are observed while studying Sturm-Liouville problems, mathematical biology, chemical systems, and reaction-diffusion equations [1, 2, 5, 37, 178, 188]. Also the differential systems with CBCs are studied while dealing with interaction problems, reaction diffusion phenomena, and Lokta-Volterra models [6, 117, 128].

The study of nonlinear CBVPs inherits complications owing to its completely different behaviours of underlying physical phenomena as compared to the uncoupled BVPs. For the sake of establishing the existence theory for nonlinear CBVPs a number of techniques are developed. The strongest of among is to establish the existence of solutions is the technique of LUSs. In this technique, the existence of solutions of the considered problems is established by modifying them logically and then the theory of differential inequalities with combination of well known existence results are applied to the modified problems.

In 1893, Picard [145] gave birth to a lower solution while looking for the solution of ODEs of the type

$$
\begin{aligned}
u_2''(t) &= g_1(t, u_2(t)), \quad t \in [c, d], \\
u_2(c) &= 0, \quad u_2(d) = 0.
\end{aligned}
\tag{0.0.1}
$$

1

He built a monotone sequence of successive approximations of the form $\alpha_{1_0} \leq \alpha_{1_1} \leq \alpha_{1_2} \leq ...$, with $\alpha_{1_0} > 0$ on (c, d) that converges to the solution of the problem (0.0.1). Further the existence of the first approximation α_{1_0} was proved that satisfies the following inequality

$$\alpha_{1_0}''(t) + g_1(t, \alpha_{1_0}(t)) > 0, \ t \in (c, d),$$

$$\alpha_{1_0}(c) = 0, \ \ \alpha_{1_0}(d) = 0.$$

Later on α_{1_0} is known by the name of a lower solution. But the major development of the method has been observed in 1931 owing to the work of Scorza Dragoni [45]. He discussed the existence results for the second-order Dirichlet BVPs of the type $u_2''(t) = g_1(t, u_2(t), u_2'(t))$, $u_2(c) = B_1$, $u_2(d) = B_2$, $t \in [c, d]$, and assumed the existence of a lower solution α_1 and an upper solution β_1, such that $\alpha_1, \beta_1 \in C^2([c, d])$ with $\alpha_1(t) \leq \beta_1(t)$ for all $t \in [c, d]$. The developed method ensured the existence of at least one solution u_1, such that $\alpha_1(t) \leq u_1(t) \leq \beta_1(t)$, for all $t \in [c, d]$. Recently, many researchers investigated the existence of solutions of BVPs with certain boundary conditions (BCs) applying LUSs approach [7, 38, 59, 118, 146, 185].

The exchange of letters between Gottfried Leibniz and Marquis de L' Hôpital in 1695 gave birth to the idea of fractional calculus. L' Hôpital inquired what would be the consequences if the order of differentiation assumed to be non-integer $\frac{1}{2}$ instead of $n \in \mathbb{N}$. During September 1695, Leibniz entertained the question of L' Hôpital with a historical sentence "this is an apparent paradox from which, one day, useful consequences will be drawn..." [129]. Many well known mathematicians had made considerable development in the field of fractional calculus, Laplace (1812), J.P.J. Fourier (1822), N.H. Abel (1823-1826), J. Liouville (1832), B. Riemann (1847), A.K. Grünwald (1867-1872), A.V. Letinkov (1868-1872), Heaviside (1892-1912), J. Hadamard (1892), H. Weyl (1917), Erdélyi (1939-1965), H. Kober (1940), and M. Riesz (1949) are the names of few. The contributions of Gorenflo and Mainardi can be seen in [71].

Integer-order derivatives have clear physical and geometrical interpretations. For instance, the first-order derivative gives slope of a tangent line, the second-order derivative elaborates the concavity of the differentiable objects. In classical physics the first-order derivative exhibits the velocity of the moving objects and the second-order derivative yields the acceleration. It is expected that being a generalization of the conventional differential calculus, the fractional calculus should have wider meanings, but still it is very miserable in this context. Owing to this deficiency, for a long time, the

subject fractional calculus could not get the attention of the researchers working on applied problems, they gave it a little respect by considering it nothing but curiosity of the mathematicians. But later on they observed the huge advantages of the fractional calculus in almost every field of applied sciences. For example in fluid dynamics a considerable advantage was examined when conventional Maxwell model [130] was replaced with a fractional order Maxwell model. Fractional order models of non-Newtonian fluids played a valuable role in the explanation of viscoelastic properties of non-Newtonian fluids [20, 46, 64].

Some material is available in the literature that explains the geometrical and physical interpretation of the fractional operators. Trustworthy geometrical and physical interpretation of the fractional integration and fractional differentiation is proposed by I. Podlubny [147]. In 1998, Torbati and Hammond [131] observed the geometrical and physical interpretation of the fractional operators in Fractal geometry by considering the fractional operators as linear filters. To explain the geometrical interpretation, they used the domino ladder network (a series of capacitors and resistors which are connected in various relationship), and Fractal Cantor's set.

In classical calculus, derivatives and integrals are uniquely computed. The similar situation exists in the case of fractional integrals. For example, many researchers like Marichev, Samko, and Kilbas [160], Podlubny [148], Oldham and Spanier [142], Miller and Ross [129] used the similar definition to compute the fractional integrals. But the situation is much different and complicated in the case of fractional order derivatives because several different competing definitions exist in the literature. For instance, a few of those include, the Riemann-Liouville approach, the Caputo approach, the Hadamard approach, the Marchaud approach, the Gränwald-Letinkov approach, the Erdélyi-Kober approach, and the Riesz-Feller approach. These existing definitions coincide only for some particular cases but in general these are not alike. Among of all these approaches to define fractional differentiation and fractional integration, the approach of Riemann-Liouville gets more attention. But the approach of Riemann-Liouville does not properly address the physics of some fractional order problems in the presence of certain boundary and/or initial conditions. Also this approach can exhibit the derivative of a constant term other than zero. To overcome this problem Caputo proposed an alternate definition of fractional derivative in 1967, and it is used in fluid dynamics in 1969 to explain the theory of viscoelasticity.

Perhaps, the first application of fractional calculus in physical problems was observed due to the work of Niels Henrik Abel (1823) while finding the solution of integer-order equation, known as tautochrone problem [8]. In this problem, the curve of an object (frictionless wire), lying in a vertical plane, was determined by using the operator $D_0^{\frac{1}{2}}$, and assuming the dependence of the time position not on the starting point.

According to the lecture notes of Ross [156], the Abel's solution of the tautochrone problem was proved so valuable because it prompted Liouville to do some logical work on the subject fractional calculus, then in 1832, the first valid and logical definition proposed by Liouville [119] on fractional differentiation was added in the literature.

During 1832 to 1855, Liouville proposed two different approaches to define the derivatives of fractional order. In the first approach the concept of nth order derivative of exponential function $D^n e^{ax} = a^n e^{ax}$ was extended to establish the idea of fractional order differentiation, As a result, the following relation was obtained

$$D^\alpha e^{ax} = a^\alpha e^{ax}. \tag{0.0.2}$$

Then a function $h(x)$ was expressed in the form of exponential series by using (0.0.2) as

$$D^\alpha h(x) = \sum_{n=0}^{\infty} b_n a_n^\alpha e^{a_n x}.$$

This was the first proposed definition of Liouville [157]. But this definition has some limitations regarding the values of α, because the series exhibits divergent behaviour for some values of α. In [47], Liouville suggested another definition to define fractional order derivative as

$$D^\alpha x^a = \frac{(-1)^\alpha \Gamma(\alpha + a)}{\Gamma(a)} x^{-\alpha - a}, \ a > 0.$$

Later on the use of this definition was seen in potential theory but it also had some limitations, because it did not deal with a wide class of functions. In 1847, Riemann proposed the definition for integral of fractional order of a function $g(x)$ by using the complementary solution of *nth* order derivative of integer order as

$$D^{-\alpha} g(x) = \frac{1}{\Gamma(\alpha)} \int_b^x (x - t)^{\alpha - 1} g(t) dt + \phi(x).$$

In 1880, Cayley commented that the complementary function of Riemann was of indeterminate form.

From 1835 to 1850, Mathematicians recorded a rich discussion on the definition of derivative of fractional order. George Peacock preferred the generalization of Lacroix instead of Liouville, while others gave importance to Liouville's definition. Augustus De Morgan asserted that both approaches dealt with a class of more general system. The debate was settled in the mid of nineteenth century. According to Harold Thayer Davis [48], "*The mathematicians at that time were looking for a plausible definition of generalized differential but, in fairness to them, one should note they lacked the tools to examine the consequences of their definition in the complex plane.*"

For many years fractional calculus had been discussed as an abstract concept of mathematics. But currently, the applications of fractional calculus can be observed in almost every field of science, including physics, bio-chemistry, engineering, image processing, bio-mathematics, and many others. It has been realized round the world that the models based on fractional derivative exhibit better results than integer-order models. Fractional derivatives inherit nonlocal nature, so it is an excellent tool to get better understanding of hereditary properties of different processes and materials. R. L. Bagley [21] presented the first Ph.D thesis on the applications of fractional calculus in Viscoelasticity models. Recently, the applications of fractional calculus have been observed in psychology to determine the time variation of emotions of mankind [9, 161]. The applications of FODPs can be seen in dynamics and control systems [10], Marine sciences and Wave dynamics [143, 149], Heat transfer models [72, 179], Diffusion processes [132], Solid mechanics [158, 171], and many more.

In the literature many monographs and books are available to study the applications and theory of fractional calculus. The comprehensive overview and historical aspects of the subject fractional calculus have been discussed in the book of K.B. Oldham and J. Spanier [142]. The detail study on FODPs and derivatives of fractional order is available in the book of K.S. Miller and B. Ross [129]. The book of I. Podlubny [147] exhibits the nature and applications of FODPs and fractional calculus. In addition many other works on the application of fractional calculus and FODPs have been added in the existing literature including the work of A. Carpinteri and F. Mainardi [39], R. Hilfer [84], A.A. Kilbas, H.M. Srivastava, J.J. Trujillo [101], J. Sabatier, O.P. Agrawal and J.A. Tenreiro Machado [162], and V. Lakshmikantham, S. Leela, and J. Vasundhara Devi [121].

The focus of attention of the second part of the thesis is the development of the reliable and efficient approximating techniques to compute the approximate solutions of the generalized class of

multi-terms FOPDEs and its coupled systems. Currently, a number of numerical techniques are available to deal with the approximate solutions of FODPs. The names of the most frequently used methods are; Differential transform method [11, 12, 60], Spectral method [22, 24, 49, 50, 51, 52], Adomian decomposition method [25, 86], Reproducing Kernal method [73], Homotopy method [100].

In this thesis, the Spectral method is employed to compute the approximate solutions of FOPDEs; it approximates the solutions in a series form. Further, this method has capability to reduce the FOPDEs into a problem of solving a system of algebraic equations which are simple in handling by any computational software and indeed makes finding the approximate solutions much easier. Different approaches are available to convert the FOPDEs into a system of algebraic equations, including collocation approach, Spectral tau approach, Galerkin approach. Galerkin approach is similar to Spectral tau approach in the way that differential equation is enforced. However, to satisfy the BCs, there is no need of the test functions. Therefore, BCs are applied in the presence of a certain supplementary set of equations [133, 150]. While finding the numerical solutions of differential equations using collocation approach, firstly, a suitable finite or discrete representation of the solutions is obtained using polynomials interpolation based on some proper nodes like well-known Gauss-Lobatto nodes. Secondly, the system of algebraic equations is obtained by discretizing the considered differential equation.

Operational matrix method is among the variants of the spectral methods [24, 50, 51]. Under this method the fractional order derivatives involved in FODPS are replaced with the operational matrices of derivatives of fractional order to obtain the system of algebraic equations. This method deals accurately with FODPs when constraints are in the form of some initial conditions. On the other hand when fractional order problems have constraints in the form of non-local and local BCs, it is not successful. This method is very much effective when the FOPDEs have one spatial variable rather than two or three spatial variables.

The arrangement of the thesis is as follows; In Chapter 1, we discuss a comprehensive introduction of coupled lower and upper solutions (CLUSs) that establishes the basis for our further works. We also include some results from Fixed Point Theory (FPT), fractional calculus, approximation theory, and matrix theory. Section 1.1 deals with the definitions of lower and upper solutions for the first-order and the second-order ordinary nonlinear differential systems along with some classical results

on the existence of solutions for periodic, anti-periodic, Dirichlet, and Neumann BVPs. Section 1.2 describes the detail introduction on the development of CLUSs. Moreover some examples are discussed in the same section to elaborate the idea of the construction of CLUSs. In Section 1.3, a brief note on the development of Nagumo conditions is given. In Section 1.4, we mention the development of the fixed point results with a historical note and also mention some basic definitions and some results from FPT that play a very important role to establish the existence of solutions for our considered nonlinear coupled BVPs. In the remaining part of the Chapter, we recall some results from fractional calculus and approximation theory. Also we discuss some properties of the orthogonal JPs that are very useful for the development of the numerical results for our later works.

Chapter 2 deals with the existence results of the first-order nonlinear ordinary coupled BVPs of the type

$$
\begin{aligned}
u_1^{'}(t) &= f_1(t, u_2(t)), \quad t \in [0,1], \\
u_2^{'}(t) &= f_2(t, u_1(t)), \quad t \in [0,1],
\end{aligned}
\tag{0.0.3}
$$

subject to nonlinear coupled boundary condition (CBC)

$$
\phi\left(u_1(0), u_2(0), u_1(1), u_2(1)\right) = (0,0),
\tag{0.0.4}
$$

where the nonlinear functions $f_1, f_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $\phi : \mathbb{R}^4 \to \mathbb{R}^2$ are continuous. In Section 2.1, we construct the method of lower and upper solutions to establish the existence results. In Section 2.3, we ensure the existence of at least one solution (u_1, u_2) of (0.0.3) such that $(\alpha_1, \alpha_2) \preceq (u_1, u_2) \preceq (\beta_1, \beta_2)$ by assuming the existence of a lower solution (α_1, α_2) and an upper solution (β_1, β_2), such that $(\alpha_1, \alpha_2) \preceq (\beta_1, \beta_2)$ on $[0,1] \times [0,1]$. In Section 2.5, the existence results are also discussed when lower and upper solutions are in reverse order, such that $(\alpha_1, \alpha_2) \succeq (\beta_1, \beta_2)$ on $[0,1] \times [0,1]$. Our chosen boundary condition (0.0.4) is very much generalized because it unifies the treatment of many different nonlinear and linear first order problems [65, 98, 126]. For instance if $\phi(x, y, z, w) = (x - C, y - D)$, then (0.0.4) implies the initial conditions, $u_1(0) = C, u_2(0) = D$, $C, D \in \mathbb{R}$. On the same fashion, if $\phi(x, y, z, w) = (x - y, z - w)$, then (0.0.4) implies the periodic conditions, $u_1(0) = u_2(0), u_1(1) = u_2(1)$. Similarly, if $\phi(x, y, z, w) = (x + y, z + w)$, then (0.0.4) implies the anti-periodic conditions, $u_1(0) = -u_2(0), u_1(1) = -u_2(1)$. On the same lines, we can establish the nonlinear-coupling among the arguments of the boundary condition (0.0.4). To satisfy the classical existence criterion for periodic, anti-periodic, and among others nonlinear BVPs,

we extend the concept of coupled lower and upper solutions explained in [66] for the two dimensional case in Sections 2.2 and 2.5.1. Our main tools to establish the existence of solutions of (0.0.3) are Arzelà-Ascolli theorem and Schauder's fixed point theorem. In Sections 2.4 and 2.5.3, we check the validity of our theoretical results by considering some examples.

In Chapter 3, we extend the findings of Chapter 2 for the nonlinear ordinary second-order coupled BVPs the type

$$
\begin{aligned}
u_1^{''}(t) &= f_1(t, u_2(t)), \quad t \in [0, 1], \\
u_2^{''}(t) &= f_2(t, u_1(t)), \quad t \in [0, 1],
\end{aligned}
\tag{0.0.5}
$$

with nonlinear CBCs

$$
\begin{aligned}
\phi \left(u_1(0), u_2(0), u_1(1), u_2(1), u_1^{'}(0), u_2^{'}(0) \right) &= (0, 0), \\
\psi \left(u_1(0), u_2(0), u_1(1), u_2(1), u_1^{'}(1), u_2^{'}(1) \right) &= (0, 0),
\end{aligned}
\tag{0.0.6}
$$

and

$$
\begin{aligned}
-u_1^{''}(t) &= f_1(t, u_2(t)), \quad t \in [0, 1], \\
-u_2^{''}(t) &= f_2(t, u_1(t)), \quad t \in [0, 1],
\end{aligned}
\tag{0.0.7}
$$

with nonlinear CBCs

$$
\begin{aligned}
\mu \left(u_1(0), u_2(0), u_1^{'}(0), u_2^{'}(0), u_1^{'}(1), u_2^{'}(1) \right) &= (0, 0), \\
\nu \left(u_1(0), u_2(0) \right) + (u_1(1), u_2(1)) &= (0, 0),
\end{aligned}
\tag{0.0.8}
$$

where $f_1, f_2 : [0, 1] \times \mathbb{R} \to \mathbb{R}$, $\phi, \psi, \mu : \mathbb{R}^6 \to \mathbb{R}^2$, and $\nu : \mathbb{R}^2 \to \mathbb{R}^2$ are continuous functions. The existence results for the second-order systems have been discussed by many researchers considering coupled and non coupled BCs [14, 15, 40, 74, 86]. In [14], a sequence of modified non-singular problem on a compact subset of $(0, 1)$ has been constructed and the existence of positive solutions of (0.0.7) has been examined by constructing a cone of positive functions, and applying the Guo-Krasnosel'skii fixed point theorem. The BCs in [14] are linear and uncoupled. But we examine the existence of solutions of (0.0.7) in the presence of nonlinear CBCs. Moreover our BCs (0.0.8) are much generalized. For example, if $\mu(j, k, l, m, n, o) = (l - n, m - o)$ and $\nu(j, k) = (-j, -k)$, then (0.0.8) implies the periodic BCs and if $\mu(j, k, l, m, n, o) = (l + n, m + 0)$ and $\nu(j, k) = (j, k)$, then (0.0.8) implies the anti-periodic BCs. Definitely, in order to obtain solutions satisfying these BCs and lying between a lower solution and an upper solution, some additional conditions are required.

For example, in the periodic case it suffices that

$$\alpha_1'(0) \geq \alpha_1'(1), \ \alpha_2'(0) \geq \alpha_2'(1), \alpha_1(1) = \alpha_1(0), \alpha_2(1) = \alpha_2(0),$$

$$\beta_1'(0) \leq \beta_1'(1), \ \beta_2'(0) \leq \beta_2'(1), \beta_1(1) = \beta_1(0), \beta_2(1) = \beta_2(0)), \tag{0.0.9}$$

and in the anti-periodic case it suffices that

$$\alpha_1'(0) \geq -\beta_1'(1), \ \alpha_2'(0) \geq -\beta_2'(1), -\alpha_1(0) = \beta_1(1), -\alpha_2(0) = \beta_2(1),$$

$$\beta_1'(0) \leq -\alpha_1'(1), \ \beta_2'(0) \leq -\alpha_2'(1), \alpha_1(1) = -\beta_1(0), \alpha_2(1) = -\beta_2(0)). \tag{0.0.10}$$

So to ensure the existence of (0.0.9) and (0.0.10), we extend the idea of CLUSs presented in [66] for a nonlinear second-order coupled problems applying some monotonicity assumptions on the arguments of nonlinear functions μ, and ν in Section 3.2.2. The monotonicity assumptions on the arguments of BCs (0.0.8) do not verify the existence criterion for the nonlinear BVPs having Dirichlet and Neumann BCs. So, to achieve the existence results for the second-order nonlinear Dirichlet and Neumann BVPs, we use the BCs (0.0.6). For instance, if $\phi(j,k,l,m,n,o) = (S - j, W - k)$ and $\psi(j,k,l,m,n,o) = (X - l, Y - m)$ with $S, W, X, Y \in \mathbb{R}$, then (0.0.6) implies the Dirichlet BCs. Similarly, if $\phi(j,k,l,m,n,o) = (n - S, o - W)$ and $\psi(j,k,l,m,n,o) = (X - n, Y - o)$, then (0.0.6) implies the Neumann BCs. Similarly, we develop the nonlinear-coupling among the arguments of BCs (0.0.6) and (0.0.8) and develop the existence of at least one solution (u_1, u_2) for the nonlinear coupled ordinary differential systems of the type (0.0.5) and (0.0.7) in Sections 3.1.3 and 3.2.3. In Section 3.1.2, we extend the idea of coupled lower and upper solutions as discussed in [68] for the second-order nonlinear coupled differential systems of the type (0.0.5) to verify the existence criterion ((0.0.11)-(0.0.12)) for Dirichlet and Neumann nonlinear BVPs under some monotonicity assumption on the arguments of the nonlinear functions ϕ and ψ

$$\alpha_1(0) \leq S \leq \beta_1(0), \ \alpha_2(0) \leq W \leq \beta_2(0),$$

$$\alpha_1(1) \leq X \leq \beta_1(1), \ \alpha_2(1) \leq Y \leq \beta_2(1), \tag{0.0.11}$$

and

$$\beta_1'(0) \leq S \leq \alpha_1'(0), \ \beta_2'(0) \leq W \leq \alpha_2'(0),$$

$$\alpha_1'(1) \leq X \leq \beta_1'(1), \ \alpha_2'(1) \leq Y \leq \beta_2'(1). \tag{0.0.12}$$

In the light of the above discussion, we claim that our results are much generalized because they unify the treatment of many nonlinear second-order coupled differential systems, like periodic, anti-periodic, Dirichlet and Neumann. In Sections 3.1.4 and 3.2.4, we check the validity of our theoretical results by taking some examples.

In Chapter 4, we extend the results of Chapter 3 by introducing the derivative terms in the nonlinear functions f_1 and f_2 and establish the existence results for nonlinear coupled systems of the type

$$
\begin{aligned}
-u_1''(t) &= f_1(t, u_1(t), u_2(t), u_1'(t), u_2'(t)), \quad t \in [0, 1], \\
-u_2''(t) &= f_2(t, u_1(t), u_2(t), u_1'(t), u_2'(t)), \quad t \in [0, 1],
\end{aligned}
\tag{0.0.13}
$$

subject to nonlinear CBCs of the type (0.0.6).

Where $f_1, f_2 : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Problems with nonlinearity dependent on derivative with certain types of BCs, like periodic, anti-periodic, Direchlet, Neumann, and multi-points has been discussed by many authors [26, 41, 68, 102, 186, 189]. In [102], the existence results for the problems having nonlinearity in first derivative with periodic and multi-point BCs were discussed using lower and upper solutions approach, the BCs used in [102] were uncoupled and linear. We extend the results for the nonlinear coupled BCs using the same approach. We discuss the existence results in the presence of the existence of a lower solution (α_1, α_2) and an upper solution (β_1, β_2), which are not necessarily constant, that is, there exist $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in C^2[0, 1] \times C^2[0, 1]$ such that

$$
\begin{aligned}
-\alpha_1''(t) &\leq f_1\left(t, \alpha_1(t), \alpha_2(t), \alpha_1'(t), \alpha_2'(t)\right), \quad t \in [0, 1], \\
-\alpha_2''(t) &\leq f_2\left(t, \alpha_1(t), \alpha_2(t), \alpha_1'(t), \alpha_2'(t)\right), \quad t \in [0, 1],
\end{aligned}
\tag{0.0.14}
$$

and

$$
\begin{aligned}
-\beta_1''(t) &\geq f_1\left(t, \beta_1(t), \beta_2(t), \beta_1'(t), \beta_2'(t)\right), \quad t \in [0, 1], \\
-\beta_2''(t) &\geq f_2\left(t, \beta_1(t), \beta_2(t), \beta_1'(t), \beta_2'(t)\right), \quad t \in [0, 1],
\end{aligned}
\tag{0.0.15}
$$

with $(\alpha_1, \alpha_2) \preceq (\beta_1, \beta_2)$ on $[0, 1] \times [0, 1]$, and for a more generalized class of BCs (0.0.6), that unify the treatment of many second-order nonlinear coupled systems of the type (0.0.13) applying some monotonicity assumptions on the arguments of nonlinear functions ϕ, and ψ, as discussed in Chapter 3. We develop the method of CLUSs for the nonlinear CBVPs by extending the idea presented in

[68] and ensure the existence of at least one solution (u_1, u_2), such that $(\alpha_1, \alpha_2) \preceq (u_1, u_2) \preceq (\beta_1, \beta_2)$ on $[0, 1] \times [0, 1]$ assuming the nonlinear functions f_1 and f_2 satisfies the Nagumo conditions relative to (α_1, α_2), (β_1, β_2) in Section 4.1.4. In section 4.1.5, we check the validity of the theoretical results by considering some examples.

In Chapter 5, we develop a reliable and an efficient numerical scheme for getting the approximate solutions of multi-terms FOPDEs and their coupled systems of the type

$$\frac{\partial^{\sigma_1} U(x,y)}{\partial x^{\sigma_1}} = a_1 \frac{\partial^{\gamma_1} U(x,y)}{\partial y^{\gamma_1}} + a_2 \frac{\partial^{\varrho_1} U(x,y)}{\partial x^{\varrho_1/2} \partial y^{\varrho_1/2}} + G_1(x,y), \tag{0.0.16}$$

corresponding to the following initial conditions with $j = 0, 1, ..., n$

$$U^{(j)}(0, y) = H_j(y), , \tag{0.0.17}$$

and a coupled system of a generalized class of FOPDEs

$$\begin{aligned}
\frac{\partial^{\sigma_1} U(x,y)}{\partial x^{\sigma_1}} &= a_1 \frac{\partial^{\gamma_1} U(x,y)}{\partial y^{\gamma_1}} + a_2 \frac{\partial^{\gamma_2} V(x,y)}{\partial x^{\gamma_2}} + a_3 \frac{\partial^{\gamma_3} V(x,y)}{\partial y^{\gamma_3}} + \\
&\quad a_4 \frac{\partial^{\varrho_4} U(x,y)}{\partial x^{\varrho_4/2} \partial y^{\varrho_4/2}} + a_5 \frac{\partial^{\varrho_3} V(x,y)}{\partial x^{\varrho_3/2} \partial y^{\varrho_3/2}} + G_1(x,y), \\
\frac{\partial^{\sigma_2} V(x,y)}{\partial x^{\sigma_2}} &= b_1 \frac{\partial^{\rho_1} V(x,y)}{\partial y^{\rho_1}} + b_2 \frac{\partial^{\rho_2} U(x,y)}{\partial x^{\rho_2}} + b_3 \frac{\partial^{\rho_3} U(x,y)}{\partial y^{\rho_1}} + \\
&\quad b_4 \frac{\partial^{\varrho_1} U(x,y)}{\partial x^{\varrho_1/2} \partial y^{\varrho_1/2}} + b_5 \frac{\partial^{\varrho_2} V(x,y)}{\partial x^{\varrho_2/2} \partial y^{\varrho_2/2}} + G_2(x,y),
\end{aligned} \tag{0.0.18}$$

corresponding to the following initial conditions with $j = 0, 1, ..., n$,

$$U^{(j)}(0, y) = H_j(y), \ V^{(j)}(0, y) = G_j(y), \tag{0.0.19}$$

where $n < \sigma_1, \sigma_2 \leq n+1$, $a_j, b_j,$ $j = 1, ..., 5$ are all real constants and $G_1(x,y), G_2(x,y), U(x,y), V(x,y) \in C([0, \Delta] \times [0, \Delta])$. Our developed numerical scheme is based on the operational matrices of integrals and derivatives of fractional order for a particular kind of polynomials, namely, the two-parametric orthogonal JPs. In Section 5.1.1, we exhibit the notion of two dimensional shifted orthogonal JPs and develop its orthogonality relationship with the aid of a weight function. In Section 5.1.3, we develop the new operational matrices of integrals and derivatives of fractional order and then in Section 5.2 these operational matrices are applied to (0.0.16) and (0.0.18) to develop a reliable and stable numerical technique for a generalized class of multi-terms FOPDEs and its coupled systems. We check the validity of our developed numerical technique in Section 5.2.3 by taking several test examples.

In Section 5.3, we extend the idea to three-dimensional shifted orthogonal JPs and developed its orthogonality relationship in Section 5.3.1. Based on the operational matrices as discussed in Chapter 5, we develop an efficient numerical method for the multi-terms high-dimensional FOPDEs. In Section 5.3.4, we discuss the application of the operational matrices for multi-terms three-dimensional heat conduction equation having mixed derivative terms of the type

$$\frac{\partial^{\sigma_1} U(t,x,y)}{\partial t^{\sigma_1}} = C_1 \frac{\partial^{\beta_1} U(t,x,y)}{\partial x^{\beta_1}} + C_2 \frac{\partial^{\beta_2} U(t,x,y)}{\partial y^{\beta_2}} + C_3 \frac{\partial^{\beta_3} U(t,x,y)}{\partial x^{\frac{\beta_3}{2}} \partial y^{\frac{\beta_3}{2}}} + G_1(t,x,y), \qquad (0.0.20)$$

corresponding to the following initial condition

$$U(0,x,y) = f(x,y). \qquad (0.0.21)$$

where C_1, C_2 and C_3 are constants, $0 < \sigma_1 \leq 1$, and $t \in [0,\Delta]$, $x \in [0,\Delta]$, $y \in [0,\Delta]$.

A significant number of test problems are chosen to check the validity of the developed algorithms included in this thesis. Exact and approximate solutions of the considered fractional order problems are compared to check the accuracy of the proposed methods. We also develop the analytical relations to attain the error bounds of approximate solutions. The convergence of the proposed methods is studied in the form of tables and plots for each test problem.

Chapter 1

Preliminaries

Throughout this thesis, we are interested in the development of the existence results of nonlinear CBVPs corresponding to nonlinear CBCs applying CLUSs approach. In the same thesis, we develop the numerical technique based on the operational matrices of integrals and derivatives of fractional order to treat with the generalized class of multi-terms FOPDEs and its coupled systems. The analytical results related to fractional calculus are taken from [129, 142, 148]. Let us begins with the idea of lower and upper solutions method.

1.1   Lower and upper solutions method

While studying the nonlinear CBVPs for the first–order and the second–order ordinary differential systems (ODSs) of the type

$$
\begin{aligned}
u_1^{'}(t) &= f_1(t, u_2(t)), \quad t \in [0, 1], \\
u_2^{'}(t) &= f_2(t, u_1(t)), \quad t \in [0, 1],
\end{aligned}
\tag{1.1.1}
$$

$$
\begin{aligned}
-u_1^{''}(t) &= f_1(t, u_2(t)), \quad t \in [0, 1], \\
-u_2^{''}(t) &= f_2(t, u_1(t)), \quad t \in [0, 1],
\end{aligned}
\tag{1.1.2}
$$

$$
\begin{aligned}
-u_1^{''}(t) &= f_1(t, u_1(t), u_2(t), u_1^{'}(t), u_2^{'}(t)), \quad t \in [0, 1], \\
-u_2^{''}(t) &= f_2(t, u_1(t), u_2(t), u_1^{'}(t), u_2^{'}(t)), \quad t \in [0, 1],
\end{aligned}
\tag{1.1.3}
$$

13

subject to certain linear or nonlinear CBCs on the compact interval $[0,1] \subseteq \mathbb{R}$, the properties of LUSs for (1.1.1),(1.1.2), and (1.1.3) are often used to investigate the existence of solutions. In LUSs approach, the existence of solutions of the considered problems is established by modifying them logically and then the theory of differential inequalities with combination of well known existence results are applied to the modified problems. The initial development of LUSs method for ODEs was observed in 1893 in the works of Picard [145]. But the major development of the method had been observed in 1931 owing to the research article [45] of Dragoni in which the existence results of the second-order Dirichlet BVPs were discussed.

LUSs method ensures the existence of at least one solution (u_1, u_2), such that $(\beta_1, \beta_2) \preceq (u_1, u_2) \preceq (\alpha_1, \alpha_2)$, satisfying $(\alpha_1, \alpha_2) \preceq (\beta_1, \beta_2)$ on $[0,1] \times [0,1]$. The case, when LUSs are in reverse order has also been discussed by many authors [66, 103, 104]. Recently, the author [17] investigated the existence of solutions of the first-order nonlinear coupled system subject to generalized nonlinear CBC, when LUSs are in reverse order, such that $(\beta_1, \beta_2) \preceq (\alpha_1, \alpha_2)$ on $[0,1] \times [0,1]$. Moreover in Chapter 2, the existence results for (1.1.1) are also discussed when LUSs are in reverse order, that is, $(\alpha_1, \alpha_2) \succeq (\beta_1, \beta_2$ on $[0,1] \times [0,1]$. Let us note the definition of LUSs

Definition 1.1.1. [Lower and upper solutions for the first-order nonlinear coupled systems] Assume f_1 and f_2 are continuous functions on $[0,1] \times \mathbb{R}$. The order pair of functions $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in C^1[0,1] \times C^1[0,1]$ are known to be LUSs of (1.1.1) respectively, if they satisfy the following set of inequalities

$$
\begin{aligned}
\alpha_1'(t) &\leq f_1\left(t, \alpha_2(t)\right), \\
\alpha_2'(t) &\leq f_2\left(t, \alpha_1(t)\right), \\
\beta_1'(t) &\geq f_1\left(t, \beta_2(t)\right), \\
\beta_2'(t) &\geq f_2\left(t, \beta_1(t)\right),
\end{aligned}
\tag{1.1.4}
$$

for all $t \in [0,1]$. If the inequalities in (1.1.4) are strict then the order pair of functions (α_1, α_2), and (β_1, β_2) are called strict LUSs respectively.

Definition 1.1.2. [Lower and upper solutions for the second-order nonlinear coupled systems] Assume f_1 and f_2 are continuous functions on $[0,1] \times \mathbb{R}$. The order pair of functions $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in C^2[0,1] \times C^2[0,1]$ are known to be LUSs of (1.1.2) respectively, if they satisfy

the following set of inequalities

$$-\alpha_1^{''}(t) \le f_1\left(t, \alpha_2(t)\right),$$

$$-\alpha_2^{''}(t) \le f_2\left(t, \alpha_1(t)\right),$$

$$-\beta_1^{''}(t) \ge f_1\left(t, \beta_2(t)\right), \qquad (1.1.5)$$

$$-\beta_2^{''}(t) \ge f_2\left(t, \beta_1(t)\right),$$

for all $t \in [0, 1]$. If the inequalities in (1.1.5) are strict then the order pair of functions (α_1, α_2), and (β_1, β_2) are called strict LUSs respectively.

Definition 1.1.3. [**Lower and upper solutions for the second-order nonlinear coupled systems with nonlinearity depending on the first derivative**] Assume f_1 and f_2 are continuous functions on $[0, 1] \times \mathbb{R}^4$. The order pair of functions $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in C^2[0, 1] \times C^2[0, 1]$ are known to be LUSs of (1.1.3) respectively, if they satisfy the following set of inequalities

$$-\alpha_1^{''}(t) \le f_1\left(t, \alpha_1(t), \alpha_2(t), \alpha_1^{'}(t), \alpha_2^{'}(t)\right),$$

$$-\alpha_2^{''}(t) \le f_2\left(t, \alpha_1(t), \alpha_2(t), \alpha_1^{'}(t), \alpha_2^{'}(t)\right),$$

$$-\beta_1^{''}(t) \ge f_1\left(t, \beta_1(t), \beta_2(t), \beta_1^{'}(t), \beta_2^{'}(t)\right), \qquad (1.1.6)$$

$$-\beta_2^{''}(t) \ge f_2\left(t, \beta_1(t), \beta_2(t), \beta_1^{'}(t), \beta_2^{'}(t)\right),$$

for all $t \in [0, 1]$. If the inequalities in (1.1.6) are strict then the order pair of functions (α_1, α_2), and (β_1, β_2) are called strict LUSs respectively.

We require certain relationships between BCs and pair of LUSs $((\alpha_1, \alpha_2), (\beta_1, \beta_2))$ to build up the specific connection at the end points 0 and 1 of $[0, 1]$. In the light of well known classical results these relationships has the following form

- In case of the first-order nonlinear coupled system for the periodic BCs, $(u_1(0) = u_1(1), u_2(0) = u_2(1))$, a pair of functions (α_1, α_2) and (β_1, β_2) satisfies

$$\alpha_1(0) \le \alpha_1(1), \ \alpha_2(0) \le \alpha_2(1)$$

$$\beta_1(0) \ge \beta_1(1), \ \beta_2(0) \ge \beta_2(1).$$

- Similarly, for the anti-periodic BCs $(u_1(0) = -u_1(1), u_2(0) = -u_2(1))$, a pair of functions

(α_1, α_2) and (β_1, β_2) satisfies

$$\alpha_1(0) \leq -\beta_1(1), \; \alpha_2(0) \leq -\beta_2(1)$$

$$\beta_1(0) \geq -\alpha_1(1), \; \beta_2(0) \geq -\alpha_2(1).$$

- In case of the second-order nonlinear coupled system for the periodic BCs, $u_1(0) = u_1(1), u_1'(0) = u_1'(1)$, and $u_2(0) = u_2(1), u_2'(0) = u_2'(1)$, a pair of functions (α_1, α_2) and (β_1, β_2) satisfies

$$\alpha_1'(0) \geq \alpha_1'(1), \; \alpha_2'(0) \geq \alpha_2'(1), \alpha_1(1) = \alpha_1(0), \alpha_2(1) = \alpha_2(0),$$

$$\beta_1'(0) \leq \beta_1'(1), \; \beta_2'(0) \leq \beta_2'(1), \beta_1(1) = \beta_1(0), \beta_2(1) = \beta_2(0)),$$

- on the same fashion, for the anti-periodic BCs, $u_1(0) = -u_1(1), u_1'(0) = -u_1'(1)$, and $u_2(0) = -u_2(1), u_2'(0) = -u_2'(1)$, a pair of functions (α_1, α_2) and (β_1, β_2) satisfies

$$\alpha_1'(0) \geq -\beta_1'(1), \; \alpha_2'(0) \geq -\beta_2'(1), -\alpha_1(0) = \beta_1(1), -\alpha_2(0) = \beta_2(1),$$

$$\beta_1'(0) \leq -\alpha_1'(1), \; \beta_2'(0) \leq -\alpha_2'(1), \alpha_1(1) = -\beta_1(0), \alpha_2(1) = -\beta_2(0)).$$

- For the Dirichlet BCs, $(u_1(0) = S, u_1(1) = X, u_2(0) = W, u_2(1) = Y)$, $X, Y, S, W \in \mathbb{R}$, we have

$$\alpha_1(0) \leq S \leq \beta_1(0), \; \alpha_2(0) \leq W \leq \beta_2(0),$$

$$\alpha_1(1) \leq X \leq \beta_1(1), \; \alpha_2(1) \leq Y \leq \beta_2(1),$$

- similarly, for the Neumann BCs, $(u_1'(0) = S, u_1'(1) = X, u_2'(0) = W, u_2'(1) = Y)$, $X, Y, S, W \in \mathbb{R}$, we have

$$\beta_1'(0) \leq S \leq \alpha_1'(0), \; \beta_2'(0) \leq W \leq \alpha_2'(0),$$

$$\alpha_1'(1) \leq X \leq \beta_1'(1), \; \alpha_2'(1) \leq Y \leq \beta_2'(1).$$

1.2 Coupled lower and upper solutions

In this section, we elaborate the idea of coupled LUSs for the first-order and the second-order BVPs subject to some specific nonlinear BCs. We discuss how the ida of couple LUSs makes possible for us to unify the treatment of many different first–order and the second–order BVPs with periodic, antiperiodic, Dirichlet, and Neumann BCs. We also discuss the fundamental results on which the main results of our thesis depend. Let us note the definitions of coupled LUSs.

Definition 1.2.1. [Coupled LUSs for the first-order BVPs][66] Assume f_1 be a nonlinear continuous function on $[0, 1] \times \mathbb{R}$, and ϕ be a nonlinear continuous function on $\mathbb{R}^2$. The functions $\alpha_1, \beta_1 \in C^1[0, 1]$ are known to be coupled LUSs for

$$u_1'(t) = f_1(t, u_1(t)), \quad t \in [0, 1], \tag{1.2.1}$$

with nonlinear boundary condition (BC) of the type

$$\phi(u_1(0), u_1(1)) = 0, \tag{1.2.2}$$

if α_1 is a lower solution and β_1 is an upper solution for (1.2.1), such that $\alpha_1 \leq \beta_1$ on $[0, 1]$, satisfying

$$\max\{\phi(\alpha_1(0), \alpha_1(1)), \phi(\alpha_1(0), \beta_1(1))\} \leq 0 \leq \min\{\phi(\beta_1(0), \beta_1(1)), \phi(\beta_1(0), \alpha_1(1))\}. \tag{1.2.3}$$

Remark 1.2.1. It is important to note that the BC (1.2.2) is very much generalized. For example, if $\phi(j, k) = j - k$, then (1.2.2) yields periodic BCs, $(u_1(0) = u_1(1))$, and if $\phi(j, k) = j + k$, then (1.2.2) implies anti-periodic BCs, $(u_1(0) = -u_1(1))$.

Remark 1.2.2. We note that (1.2.3) generalizes the classical results for periodic and anti-periodic BCs by applying the monotonicity assumptions on the arguments of (1.2.2). For instance, if ϕ is monotone decreasing in the second argument then (1.2.3) satisfies the classical existence results, $\alpha_1(0) \leq \alpha_1(1)$, and $\beta_1(0) \geq \beta_1(1)$ for the periodic BVPs. On the same fashion, if ϕ is monotone increasing in the second arguments then (1.2.3) verifies the classical existence results, $\alpha_1(0) \leq -\beta_1(1)$, and $\beta_1(0) \geq -\alpha_1(1)$ for the anti-periodic BVPs.

Remark 1.2.3. Coupled LUSs for the problem (1.2.1)-(1.2.2) can be also defined when LUSs are in reverse order, such that, $\alpha_1 \geq \beta_1$ on $[0, 1]$. In this case, (1.2.3) becomes

$$\max\{\phi(\alpha_1(0), \alpha_1(1)), \phi(\beta_1(0), \alpha_1(1))\} \leq 0 \leq \min\{\phi(\beta_1(0), \beta_1(1)), \phi(\alpha_1(0), \beta_1(1))\}. \tag{1.2.4}$$

(1.2.4) also generalizes the classical existence results for the periodic and anti-periodic BVPs by applying the monotonicity assumptions on the arguments of (1.2.2), when LUSs are in reverse order, such that, $\alpha_1 > \beta_1$ on $[0, 1]$.

Example 1.2.1. *Consider the following boundary value problem (BVP)*

$$u_1'(t) = -2u_1(t) + \lambda \sin(10t), \quad t \in [0, 1], \tag{1.2.5}$$

with two-point nonlinear boundary condition (NLBC), defined as

$$\phi\left(u_1(0), u_1(1)\right) = u_1^3(0) - u_1^3(1), \tag{1.2.6}$$

where λ is a positive integer. Let $\alpha_1(t) = -2\lambda$, and $\beta_1(t) = 2\lambda$, with $\alpha_1(t) < \beta_1(t)$, for all $t \in [0,1]$. Obviously, -2λ, and 2λ are LUSs respectively of (1.2.5), because

$$\alpha_1'(t) = 0 \le f_1(t, \alpha_1(t)) = -2(\alpha_1(t)) + \lambda \sin(10t), \quad t \in [0,1]$$
$$= 4\lambda + \lambda \sin(10t),$$
$$\beta_1'(t) = 0 \ge f_1(t, \beta_1(t)) = -2(\beta_1(t)) + \lambda \sin(10t), \quad t \in [0,1] \tag{1.2.7}$$
$$= -4\lambda + \lambda \sin(10t).$$

Moreover, in the light of (1.2.3) and (1.2.7), -2λ, and 2λ are coupled LUSs respectively of (1.2.5)-(1.2.6), because, if ϕ is monotone increasing in the second argument, then (1.2.3) yields

$$\phi\left(\alpha_1(0), \beta_1(1)\right) = \alpha_1^3(0) - \beta_1^3(1)$$
$$= -8\lambda^3 - 8\lambda^3$$
$$< 0,$$
$$\phi\left(\beta_1(0), \alpha_1(1)\right) = \beta_1^3(0) - \alpha_1^3(1)$$
$$= 8\lambda^3 + 8\lambda^3$$
$$> 0.$$

Similarly,

if ϕ is monotone decreasing in the second argument, then (1.2.3) implies

$$\phi\left(\alpha_1(0), \alpha_1(1)\right) = \alpha_1^3(0) - \alpha_1^3(1)$$
$$= -8\lambda^3 + 8\lambda^3$$
$$= 0,$$
$$\phi\left(\beta_1(0), \beta_1(1)\right) = \beta_1^3(0) - \beta_1^3(1)$$
$$= 8\lambda^3 - 8\lambda^3$$
$$= 0.$$

Example 1.2.2. *Consider the following BVP*

$$u_1'(t) = 4u_1^5(t) + \lambda \cos(5t), \quad t \in [0,1], \tag{1.2.8}$$

with two-point NLBC defined as

$$\phi\left(u_1(0), u_1(1)\right) = u_1(1)u_1(0) - u_1^4(0)u_1(1), \tag{1.2.9}$$

where λ is a positive integer. Let $\alpha_1(t) = 3\lambda$, and $\beta_1(t) = -3\lambda$, with $\alpha_1(t) > \beta_1(t)$, for all $t \in [0, 1]$.
Obviously, 3λ, and -3λ are the LUSs respectively of (1.2.8), because

$$\alpha_1'(t) = 0 < f_1(t, \alpha_1(t)) = 4\left(\alpha_1(t)\right)^5 + \lambda\cos\left(5t\right), \quad t \in [0, 1]$$

$$= 972\lambda^5 + \lambda\cos\left(5t\right),$$

$$\beta_1'(t) = 0 > f_1(t, \beta_1(t)) = 4\left(\beta_1(t)\right)^5 + \lambda\cos\left(5t\right), \quad t \in [0, 1] \tag{1.2.10}$$

$$= -972\lambda^5 + \lambda\cos\left(5t\right).$$

Moreover, in the light of (1.2.4) and (1.2.10), 3λ, and -3λ are coupled LUSs respectively of (1.2.8)-
(1.2.9), because, if ϕ is monotone increasing in the first argument, then (1.2.4) yields

$$\phi\left(\alpha_1(0), \alpha_1(1)\right) = \alpha_1(1)\alpha_1(0) - \alpha_1^4(0)\alpha_1(1)$$

$$= 9\lambda^2 - 243\lambda^5$$

$$< 0,$$

$$\phi\left(\beta_1(0), \beta_1(1)\right) = \beta_1(1)\beta_1(0) - \beta_1^4(0)\beta_1$$

$$= 9\lambda^2 + 243\lambda^5$$

$$> 0.$$

Similarly,

if ϕ is monotone decreasing in the first argument, then (1.2.4) implies

$$\phi\left(\beta_1(0), \alpha_1(1)\right) = \alpha_1(1)\beta_1(0) - \beta_1^4(0)\alpha_1(1)$$

$$= -9\lambda^2 - 243\lambda^5$$

$$< 0,$$

$$\phi\left(\alpha_1(0), \beta_1(1)\right) = \beta_1(1)\alpha_1(0) - \alpha_1^4(0)\beta_1(1)$$

$$= -9\lambda^2 + 243\lambda^5$$

$$> 0.$$

Definition 1.2.2. [Coupled LUSs for the second-order BVPs][68] Assume f_1 be a nonlinear
continuous function on $[0, 1] \times \mathbb{R}$, and ϕ, ψ are the nonlinear continuous functions on $\mathbb{R}^3$. The func-
tions $\alpha_1, \beta_1 \in C^2[0, 1]$ are known to be coupled LUSs for

$$u_1''(t) = f_1\left(t, u_1(t)\right), \quad t \in [0, 1], \tag{1.2.11}$$

with nonlinear BCs

$$\phi\left(u_1(0), u_1(1), u_1'(0)\right) = 0,$$
$$\psi\left(u_1(0), u_1(1), u_1'(1)\right) = 0, \tag{1.2.12}$$

if α_1 is a lower solution and β_1 is an upper solution for (1.2.11), with $\alpha_1 \leq \beta_1$ on $[0, 1]$, satisfying

$$\max\{\phi\left(\beta_1(0), \beta_1(1), \beta_1'(0)\right), \phi\left(\beta_1(0), \alpha_1(1), \beta_1'(0)\right)\} \leq 0$$

$$\min\{\phi\left(\alpha_1(0), \alpha_1(1), \alpha_1'(0)\right), \phi\left(\alpha_1(0), \beta_1(1), \alpha_1'(0)\right)\} \geq 0,$$

and $\tag{1.2.13}$

$$\max\{\psi\left(\beta_1(0), \beta_1(1), \beta_1'(1)\right), \psi\left(\alpha_1(0), \beta_1(1), \beta_1'(1)\right)\} \leq 0$$

$$\min\{\psi\left(\alpha_1(0), \alpha_1(1), \alpha_1'(1)\right), \psi\left(\beta_1(0), \alpha_1(1), \alpha_1'(1)\right)\} \geq 0.$$

Remark 1.2.4. It is worth mentioning that (1.2.12) generalizes most of the usual linear and nonlinear BCs. For example, if $\phi\left(j, k, l\right) = C - j$, and $\psi\left(j, k, l\right) = D - k$, then (1.2.12) gives Dirichlet BCs, $\left(u_1(0) = C, u_1(1) = D\right)$. Similarly, if $\phi\left(j, k, l\right) = l - C$, and $\psi\left(j, k, l\right) = D - l$, then (1.2.12) gives Neumann BCs, $\left(u_1'(0) = C, u_1'(1) = D\right)$.

Remark 1.2.5. We observe that (1.2.13) generalizes the classical existence results for Dirichlet and Neumann BVPs by applying the monotonicity assumptions on the arguments of (1.2.12). For instance, if ϕ is monotone increasing in the second argument and ψ is monotone increasing in the first argument then (1.2.13) satisfies the classical existence results, $\alpha_1(0) \leq C \leq \beta_1(0)$, and $\alpha_1(1) \leq D \leq \beta_1(1)$ for the Dirichlet BVPs. On the same fashion, if ϕ is monotone increasing in the second argument and ψ is monotone increasing in the first argument then (1.2.13) satisfies the classical existence results, $\beta_1'(0) \leq C \leq \alpha_1'(0)$, and $\alpha_1'(1) \leq D \leq \beta_1'(1)$ for the Neumann BVPs.

Remark 1.2.6. (1.2.12) and (1.2.13) do not allow us to generalize the classical existence results for periodic and anti-periodic BVPs. But if we replace (1.2.12) with the following nonlinear BCs

$$\phi\left(u_1(0), u_1'(0), u_1'(1)\right) = 0,$$
$$\psi\left(u_1(0)\right) + u_1(1) = 0, \tag{1.2.14}$$

then (1.2.14) allows us to verify the existence criterion for periodic and anti-periodic BVPs under the monotonicity assumptions on the arguments of (1.2.14). For this case, (1.2.13) becomes

$$\max\{\phi\left(\beta_1(0), \beta_1'(0), \beta_1'(1)\right), \phi\left(\beta_1(0), \beta_1'(0), \alpha_1'(1)\right)\} \leq 0$$

$$\min\{\phi\left(\alpha_1(0), \alpha_1'(0), \alpha_1'(1)\right), \phi\left(\alpha_1(0), \alpha_1'(0), \beta_1'(1)\right)\} \geq 0,$$

and $\hspace{6cm}$ (1.2.15)

$$\max\{\psi\left(\beta_1(0)\right), \psi\left(\alpha_1(0)\right)\} + \alpha_1(1) = 0$$

$$\min\{\psi\left(\alpha_1(0)\right), \psi\left(\beta_1(0)\right)\} + \beta_1(1) = 0.$$

Example 1.2.3. *Consider the following BVP*

$$u_1''(t) = u_1^3(t) + \cos\left(t\right), \quad t \in [0,1], \hspace{3cm} (1.2.16)$$

with two-point nonlinear BCs defined as

$$\phi\left(u_1(0), u_1(1), u_1'(0)\right) = u_1(0)u_1(1) - (u_1(1)u_1'(0))^3,$$

$\hspace{9cm}$ (1.2.17)

$$\psi\left(u_1(0), u_1(1), u_1'(1)\right) = u_1(0)u_1'(1) - (u_1(1)u_1'(0))^5.$$

Let $\alpha_1(t) = \sin(t) - 6$, and $\beta_1(t) = \sin(t) + 6$, with $\alpha_1(t) < \beta_1(t)$, for all $t \in [0,1]$. Obviously, $\sin(t) - 6$, and $\sin(t) + 6$ are LUSs respectively of (1.2.16), because

$$\alpha_1''(t) = -\sin(t) > f_1(t, \alpha_1(t)) = (\alpha_1(t))^3 + \cos\left(t\right), \quad t \in [0,1]$$

$$= (\sin(t) - 6)^3 + \cos(t)$$

$$< -\sin(t),$$

$\hspace{10cm}$ (1.2.18)

$$\beta_1''(t) = -\sin(t) < f_1(t, \beta_1(t)) = (\beta_1(t))^3 + \cos\left(t\right), \quad t \in [0,1]$$

$$= (\sin(t) + 6)^3 + \cos(t)$$

$$> -\sin(t).$$

Moreover, in the light of (1.2.13) and (1.2.18), $\sin(t)-6$, and $\sin(t)+6$ are coupled LUSs respectively

of (1.2.16)-(1.2.17), because, if ϕ is monotone increasing in the second argument, then (1.2.13) yields

$$\phi\left(\beta_1(0), \beta_1(1), \beta_1'(0)\right) = \beta_1(0)\beta_1(1) - \left(\beta_1(1)\beta_1'(0)\right)^3$$

$$= 6\sin(1) + 36 - \left(\sin(1) + 6\right)^3$$

$$- 279.17$$

$$< 0,$$

$$\phi\left(\alpha_1(0), \alpha_1(1), \alpha_1'(0)\right) = \alpha_1(0)\alpha_1(1) - \left(\alpha_1(1)\alpha_1'(0)\right)^3$$

$$= 6\sin(1) - 36 - \left(\sin(1) - 6\right)^3$$

$$= 106.32$$

$$> 0.$$

Similarly,

if ϕ is monotone increasing in the first argument, then (1.2.13) implies

$$\psi\left(\beta_1(0), \beta_1(1), \beta_1'(1)\right) = \beta_1(0)\beta_1'(1) - \left(\beta_1(1)\beta_1'(0)\right)^5$$

$$= 6\cos(1) - \left(\sin(1) + 6\right)^5$$

$$- 14984.89$$

$$< 0,$$

$$\phi\left(\alpha_1(0), \alpha_1(1), \alpha_1'(1)\right) = \alpha_1(0)\alpha_1'(1) - \left(\alpha_1(1)\alpha_1'(0)\right)^5$$

$$= -6 - \left(\sin(1) - 6\right)^5$$

$$= 3646.83$$

$$> 0.$$

Example 1.2.4. *Consider the following BVP*

$$u_1''(t) = 5u_1'(t) + \left(u_1(t) + 1\right)^2 - \sin^2\left(\pi t\right), \quad t \in [0, 1], \tag{1.2.19}$$

with two-point nonlinear BCs defined as

$$\phi\left(u_1(0), u_1(1), u_1'(0)\right) = \left(u_1(0)\sin\left(u_1(0)\right) + \cos\left(u_1'(0)\right)u_1(1)\right),$$

$$\psi\left(u_1(0), u_1(1), u_1'(1)\right) = \left(u_1(0)\tan\left(u_1(0)\right) + \cos\left(u_1'(1)\right)u_1(1)\right). \tag{1.2.20}$$

Let $\alpha_1(t) = -t^2 - t$, and $\beta_1(t) = t^2 + t$, with $\alpha_1(t) \leq \beta_1(t)$, for all $t \in [0, 1]$. Obviously, $-t^2 - t$, and

$t^2 + t$ *are the LUSs respectively of* (1.2.19)*, because*

$$\alpha_1''(t) = -2 > f_1(t, \alpha_1(t), \alpha_1'(t)) = 5\alpha_1'(t) + (\alpha_1(t) + 1)^2 - \sin^2(\pi t), \quad t \in [0, 1]$$

$$= -5(2t + 1) + (t^2 + t - 1)^2 - \sin^2(\pi t)$$

$$= t^4 + 2t^3 - t^2 - 12t - (4 + \sin^2(\pi t))$$

$$< -2, \tag{1.2.21}$$

$$\beta_1''(t) = 2 < f_1(t, \beta_1(t), \beta_1'(t)) = 5\beta_1'(t) + (\beta_1(t) + 1)^2 - \sin^2(\pi t), \quad t \in [0, 1]$$

$$= 5(2t + 1) + (t^2 + t + 1)^2 - \sin^2(\pi t)$$

$$> 2.$$

Moreover, in the light of (1.2.13) *and* (1.2.21)*,* $-t^2 - t$*, and* $t^2 + t$ *are coupled LUSs respectively of* (1.2.19)-(1.2.20)*, because, if* ϕ *is monotone increasing in the second argument, then* (1.2.13) *yields*

$$\phi(\beta_1(0), \beta_1(1), \beta_1'(0)) = (\beta_1(0)\sin(\beta_1(0)) + \cos(\beta_1'(0))\beta_1(1))$$

$$= -2\cos(1)$$

$$- 1.0806$$

$$< 0,$$

$$\phi(\alpha_1(0), \alpha_1(1), \alpha_1'(0)) = (\alpha_1(0)\sin(\alpha_1(0)) + \cos(\alpha_1'(0))\alpha_1(1))$$

$$= 2\cos(1)$$

$$= 1.0806$$

$$> 0.$$

Similarly,

if ϕ *is monotone increasing in the first argument, then* (1.2.13) *implies*

$$\psi(\beta_1(0), \beta_1(1), \beta_1'(1)) = (\beta_1(0)\tan(\beta_1(0)) + \cos(\beta_1'(1))\beta_1(1))$$

$$= 2\cos(3)$$

$$- 1.98$$

$$< 0,$$

and

$$\phi\left(\alpha_1(0), \alpha_1(1), \alpha_1'(1)\right) = \left(\alpha_1(0)\tan\left(\alpha_1(0)\right) + \cos\left(\alpha_1'(1)\right)\alpha_1(1)\right)$$

$$= -2\cos(3)$$

$$= 1.98$$

$$> 0.$$

We need the following lemma [66] for the extension of our later work.

Lemma 1.2.1. *Let* $L : C^1[0,1] \to C_0^1[0,1] \times \mathbb{R}$ *be defined by*

$$[Lu_1](t) = \left(u_1(t) - u_1(0) + \lambda\int_0^t u_1(s)\mathrm{d}s, au_1(0) + bu_1(1)\right), \tag{1.2.22}$$

where $\lambda, a,$ *and* b *are real constants with* $\lambda > 0,$ *such that*

$$\left(a + be^{-\lambda}\right) \neq 0,$$

and

$$C_0^1[0,1] = \left\{w \in C^1[0,1] : w(0) = 0\right\}.$$

Then L^{-1} *exists and is continuous and defined by*

$$[L^{-1}(y,\gamma)](t) = e^{-\lambda t}A + y(t) - \lambda\int_0^t e^{\lambda(s-t)}y(s)\mathrm{d}s, \tag{1.2.23}$$

with

$$A = \frac{\gamma + \lambda b\int_0^1 e^{\lambda(s-1)}y(s)\,\mathrm{d}s - by(1)}{a + be^{-\lambda}}.$$

Proof. Choose

$$y(t) = u_1(t) - u_1(0) + \lambda\int_0^t u_1(s)\mathrm{d}s, \tag{1.2.24}$$

and

$$\gamma = au_1(0) + bu_1(1). \tag{1.2.25}$$

In the light of (1.2.24)-(1.2.25), (1.2.22) can also be written as

$$[Lu_1](t) = (y(t),\gamma). \tag{1.2.26}$$

Differentiating (1.2.24) w.r.t. t, we have

$$y'(t) = u_1'(t) + \lambda u_1(t). \tag{1.2.27}$$

Multiplying (1.2.27) with integrating factor $e^{\lambda t}$, we have

$$e^{\lambda t} y'(t) = \left(u_1(t) e^{\lambda t} \right)', \tag{1.2.28}$$

then after integrating and taking the limits of integration from 0 to t, (1.2.28) becomes

$$u_1(t) = u_1(0) e^{-\lambda t} + y(t) - \lambda \int_0^t e^{\lambda(s-t)} y(s) ds, \tag{1.2.29}$$

$u_1(0)$ can easily be determined with the help of (1.2.25) as

$$\gamma = \left(a + b e^{-\lambda} \right) u_1(0) + by(1) - b\lambda \int_0^1 e^{\lambda(s-1)} y(s) ds,$$

which further implies

$$u_1(0) = \frac{\gamma + b\lambda \int_0^1 e^{\lambda(s-1)} y(s) ds - by(1)}{a + b e^{-\lambda}}, \quad a + b e^{-\lambda} \neq 0, \tag{1.2.30}$$

for simplicity of notation, let

$$A = \frac{\gamma + b\lambda \int_0^1 e^{\lambda(s-1)} y(s) ds - by(1)}{a + b e^{-\lambda}}, \quad a + b e^{-\lambda} \neq 0. \tag{1.2.31}$$

Using (1.2.31) in (1.2.29), we have

$$u_1(t) = A e^{-\lambda t} + y(t) - \lambda \int_0^t e^{\lambda(s-t)} y(s) ds. \tag{1.2.32}$$

Equation 1.2.26 implies

$$[L^{-1}(y, \gamma)](t) = u_1(t). \tag{1.2.33}$$

Hence, (1.2.31) proves the result. $\qquad\square$

The extension of Lemma 1.2.1, is the following

Lemma 1.2.2. *Let* $L : C^2[0,1] \to C_0^2[0,1] \times \mathbb{R} \times \mathbb{R}$ *be defined by*

$$[Lu_1](t) = \left(u_1'(t) - u_1'(0) - \lambda \int_0^t u_1(s) ds, \, au_1(0) + bu_1(1), \, cu_1(0) + du_1(1) \right), \tag{1.2.34}$$

where $\lambda > 0$ *is a real constant. Moreover,* d, c, b *and* a *are also treated as real constants, such that*

$$(ad - bc) \left(e^{-\sqrt{\lambda}} - e^{\sqrt{\lambda}} \right) \neq 0,$$

and here

$$C_0^2[0,1] = \left\{ w_1 \in C^2[0,1] : x(0) = 0 \right\}.$$

Then L^{-1} exists and is continuous and defined by

$$
\begin{aligned}
[L^{-1}(y, \gamma, \delta)](t) \;=\; & \left(A e^{\sqrt{\lambda} t} + B e^{-\sqrt{\lambda} t} + \frac{1}{2} \int_0^t e^{\sqrt{\lambda}(t-s)} y(s)\, \mathrm{d}s \right. \\
& \left. - \frac{1}{2} \int_0^t e^{\sqrt{\lambda}(s-t)} y(s)\, \mathrm{d}s \right),
\end{aligned}
$$

with

$$
\begin{aligned}
A \;=\; & \frac{1}{(ad-bc)\left(e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}\right)} \left(2\delta \left(a + be^{-\sqrt{\lambda}}\right) - d\left(a + be^{-\sqrt{\lambda}}\right) \right. \\
& \int_0^1 e^{\sqrt{\lambda}(1-s)} y(s)ds + d\left(a + be^{-\sqrt{\lambda}}\right) \int_0^1 e^{\sqrt{\lambda}(s-1)} y(s)ds \\
& - 2\gamma\left(c + de^{-\sqrt{\lambda}}\right) + b\left(c + de^{-\sqrt{\lambda}}\right) \int_0^1 e^{\sqrt{\lambda}(1-s)} y(s)ds \\
& \left. - b\left(c + de^{-\sqrt{\lambda}}\right) \int_0^1 e^{\sqrt{\lambda}(s-1)} y(s)ds \right),
\end{aligned}
$$

and

$$
\begin{aligned}
B \;=\; & \frac{1}{(ad-bc)\left(e^{-\sqrt{\lambda}} - e^{\sqrt{\lambda}}\right)} \left(2\delta \left(a + be^{\sqrt{\lambda}}\right) - d\left(a + be^{\sqrt{\lambda}}\right) \right. \\
& \int_0^1 e^{\sqrt{\lambda}(1-s)} y(s)ds + d\left(a + be^{\sqrt{\lambda}}\right) \int_0^1 e^{\sqrt{\lambda}(s-1)} y(s)ds \\
& - 2\gamma\left(c + de^{\sqrt{\lambda}}\right) + b\left(c + de^{\sqrt{\lambda}}\right) \int_0^1 e^{\sqrt{\lambda}(1-s)} y(s)ds \\
& \left. - b\left(c + de^{\sqrt{\lambda}}\right) \int_0^1 e^{\sqrt{\lambda}(s-1)} y(s)ds \right).
\end{aligned}
$$

Proof. Choose

$$
y(t) = u_1'(t) - u_1'(0) - \lambda \int_0^t u_1(s)\, \mathrm{d}s, \tag{1.2.35}
$$

$$
\gamma = a u_1(0) + b u_1(1), \tag{1.2.36}
$$

$$
\delta = c u_1(0) + d u_1(1), \tag{1.2.37}
$$

In the light of (1.2.35)-(1.2.37), (1.2.34) can also be written as

$$
[L(u_1)](t) = (y(t), \gamma, \delta). \tag{1.2.38}
$$

After differentiating the Equation 1.2.35, we have

$$
y'(t) = u_1''(t) - \lambda u_1(t), \quad \lambda > 0. \tag{1.2.39}
$$

The general solution of (1.2.39) can be easily determined using variation of parameters technique along with integration by parts and taking limits of integration from 0 to t, we have

$$u_1(t) = Ae^{\sqrt{\lambda}t} + Be^{-\sqrt{\lambda}t} + \frac{1}{2}\int_0^t e^{\sqrt{\lambda}(t-s)}y(s)\mathrm{d}s + \frac{1}{2}\int_0^t e^{\sqrt{\lambda}(s-t)}y(s)\mathrm{d}s. \qquad (1.2.40)$$

A and B can be easily determined with the help of (1.2.36) and (1.2.37) as

$$\gamma = \left(a + be^{\sqrt{\lambda}}\right)A + \left(a + be^{-\sqrt{\lambda}}\right)B + \frac{b}{2}\left(\int_0^1 e^{\sqrt{\lambda}(1-s)}y(s)\mathrm{d}s + e^{\sqrt{\lambda}(s-1)}y(s)\mathrm{d}s\right),$$

$$\delta = \left(c + de^{\sqrt{\lambda}}\right)A + \left(c + de^{-\sqrt{\lambda}}\right)B + \frac{d}{2}\left(\int_0^1 e^{\sqrt{\lambda}(1-s)}y(s)\mathrm{d}s + e^{\sqrt{\lambda}(s-1)}y(s)\mathrm{d}s\right). \qquad (1.2.41)$$

Solving (1.2.41), we have

$$
\begin{aligned}
A \;=\; & \frac{1}{(ad-bc)\left(e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}\right)}\left(2\delta\left(a + be^{-\sqrt{\lambda}}\right) - d\left(a + be^{-\sqrt{\lambda}}\right)\right. \\
& \int_0^1 e^{\sqrt{\lambda}(1-s)}y(s)ds + d\left(a + be^{-\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(s-1)}y(s)ds \\
& - \; 2\gamma\left(c + de^{-\sqrt{\lambda}}\right) + b\left(c + de^{-\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(1-s)}y(s)ds \\
& - \; b\left(c + de^{-\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(s-1)}y(s)ds\left.\right),
\end{aligned}
$$

and

$$
\begin{aligned}
B \;=\; & \frac{1}{(ad-bc)\left(e^{-\sqrt{\lambda}} - e^{\sqrt{\lambda}}\right)}\left(2\delta\left(a + be^{\sqrt{\lambda}}\right) - d\left(a + be^{\sqrt{\lambda}}\right)\right. \\
& \int_0^1 e^{\sqrt{\lambda}(1-s)}y(s)ds + d\left(a + be^{\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(s-1)}y(s)ds \\
& - \; 2\gamma\left(c + de^{\sqrt{\lambda}}\right) + b\left(c + de^{\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(1-s)}y(s)ds \\
& - \; b\left(c + de^{\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(s-1)}y(s)ds\left.\right). \qquad (1.2.42)
\end{aligned}
$$

(1.2.38) can also be written as

$$u_1(t) = [L{-}1\,(y, \gamma, \delta)](t). \qquad (1.2.43)$$

Hence (1.2.42) proves the required result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

A fundamental result to ensure the existence of at least one solution u_1 of (1.2.1) corresponding to generalized nonlinear BCs (1.2.2) under the approach of coupled LUSs is the following (see [66])

Theorem 1.2.3. *Assume that α_1 and β_1 are the CLUSs for the BVP (1.2.1)-(1.2.2). In addition, suppose that the functions*

$$\phi_{\alpha_1}(x) := \phi\left(\alpha_1(0), x\right),$$

$$\phi_{\beta_1}(x) := \phi\left(\beta_1(0), x\right),$$

are monotone on $[\alpha_1(1), \beta_1(1)]$, then (1.2.1)-(1.2.2) has at least one solution, $u_1 \in [\![\alpha_1, \beta_1]\!]$.

Example 1.2.5. *Consider the following BVP*

$$u_1'(t) = -2u_1(t) + \lambda \sin(10t), \quad t \in [0, 1], \tag{1.2.44}$$

with two-point NLBC, defined as

$$\phi\left(u_1(0), u_1(1)\right) = u_1^3(0) - u_1^3(1), \tag{1.2.45}$$

where λ is a positive integer. Let $\alpha_1(t) = -2\lambda$, and $\beta_1(t) = 2\lambda$, with $\alpha_1(t) < \beta_1(t)$, for all $t \in [0, 1]$. Obviously, -2λ, and 2λ are the LUSs respectively of (1.2.44) as already been varified in Example 1.2.5

Moreover, it has already been varified in Example 1.2.5, that -2λ, and 2λ are the CLUSs respectively of (1.2.44)-(1.2.45).

Further the functions, $(\phi_{\alpha_1}(x), \phi_{\beta_1}(x) : \mathbb{R}^2 \to \mathbb{R})$ are monotone non-decreasing on $[\alpha_1(1), \beta_1(1)] = [-2\lambda, 2\lambda]$ because for every $x_1 \leq x_2$ in $[-2\lambda, 2\lambda]$, we have

$$\phi_{\alpha_1}(x_1) \preceq \phi_{\alpha_1}(x_2)$$

$$\phi(\alpha_1(0), x_1) \preceq \phi(\alpha_1(0), x_2)$$

$$x_1^3 - \alpha_1^3(0) \leq x_2^3 - \alpha_1^3(0)$$

$$x_1 \leq x_2,$$

and

$$\phi_{\beta_1}(x_1) \preceq \phi_{\beta_1}(x_2)$$

$$\phi(\beta_1(0), x_1) \preceq \phi(\beta_1(0), x_2)$$

$$x_1^3 - \beta_1^3(0) \leq x_2^3 - \beta_1^3(0)$$

$$x_1 \leq x_2.$$

Hence according to the Theorem 1.2.3, (1.2.44)-(1.2.45) has at least one solution $u_1 \in [\![\alpha_1, \beta_1]\!]$.

1.3 Nagumo condition

In this section, we talk about the existence results of a nonlinear BVPs to the case when the nonlinear term is of the form $f_1(t, u_1, u_1')$, that is also depends on u_1'. Existence of well-ordered LUSs are enough to ensure the existence of solutions of a nonlinear BVPs when nonlinear term $f_1(t, u_1, u_1')$ is independent of u_1'. But the existence of well–ordered LUSs is not sufficient to ensure the existence of solutions for the case when nonlinearity f_1 also depends on u_1'. For example, $u_1(t) := 4 - \sqrt{4 - t}$ is a solution of a nonlinear differential equation $u_1''(t) = 2(u_1'(t))^3$ on $[0, 4)$. Clearly, $u_1(t) := 4 - \sqrt{4 - t}$ is bounded on $[0, 4)$ but its derivative is unbounded on $[0, 4)$. This is due to the fact that the nonlinear term $f_1(t, u_1, u_1') = 2(u_1'(t))^3$ grows very rapidly with respect to the gradient of the solution. Moreover, in 1954, Nagumo [137] took an example of Dirichlet problem and proved that the considered problem had no solution although the ordered lower and upper solutions exist. Later on Habets and Pouse [87] extended the results of Nagumo to the periodic and separated BCs and proved that the method of LUSs is not valid without considering Nagumo conditions. Therefore to ensure the existence of a solution, we need to find a prior bound on the gradient of a solution. For this a very fruitful assumption is used, that is, for nonlinear BVPs, the nonlinearity f_1 grows not faster than quadratically with respect to the gradient. Consider the following differential equation on the interval $(0, 1)$

$$u_1''(t) = f_1(t, u_1, u_1'), \quad t \in (0, 1). \tag{1.3.1}$$

The nonlinear term $f_1(t, u_1, u_1')$ in the presence of Nagumo condition exhibits the maximum growth rate with respect to the gradient of the solution that is quadratic. We need to estimate u_1' on $(0, 1)$ to ensure the existence of solution of (1.3.1) in the presence of some BCs. For this we will impose very useful growth condition which is the result of Nagumo [138] and Bernstein[28] works, such that, for all $t \in (0, 1)$ and $u_1' \in \mathbb{R}$, we have

$$|f_1(t, u_1, u_1')| \le c(u_1)(1 + |u_1'|^2). \tag{1.3.2}$$

To observe that (1.3.2) gives boundness of $|u_1'|$, assuming boundness of $|u_1|$. For this assume $|u_1(t)| \le N$, for all $t \in (0, 1)$, s_0 be the extremum point of $u_1(t)$ and $s_1 \in (0, 1)$ with $s_0 < s_1$. If $u_1'(t) \ge 0$ on

$[s_0, s_1]$, then in the light of (1.3.2), we have

$$\frac{u_1''(t)u_1'(t)}{1 + |u_1'(t)|^2} \le c(u_1(t))(u_1'(t)), \quad t \in [s_0, s_1], \tag{1.3.3}$$

which further implies that

$$\frac{d}{dt}\frac{1}{2}\ln(1 + u_1'^2(t)) - \frac{d}{dt}C(u_1) \le 0, \quad t \in [s_0, s_1], \tag{1.3.4}$$

where $C(u_1) = \int_0^{u_1} c(s)ds$. So, $\frac{1}{2}\ln(1 + u_1'^2(t)) - C(u_1)$ is decreasing on $[s_0, s_1]$. Hence

$$\frac{1}{2}\ln(1 + u_1'^2(t)) \le C(u_1) - C(u_1(s_0)). \tag{1.3.5}$$

(1.3.5) implies that u_1' is bounded on $[s_0, s_1]$. If $u_1'(t) \le 0$ on $[s_2, s_3]$, where $0 \le s_2 \le s_3 \le 1$. Then (1.3.2) implies

$$u_1''(t) \ge -c(u_1(t))(1 + |u_1'(t)|^2). \tag{1.3.6}$$

Hence

$$u_1''(t)u_1'(t) \le -c(u_1(t))(1 + |u_1'(t)|^2)u_1'(t). \quad t \in [s_2, s_3]. \tag{1.3.7}$$

Hence u_1' is bounded on $(0, 1)$. The important point is that this argument does not depend on the BCs. If Nagumo condition is violated then u_1' need not be bounded. For example, the following BVP [105],

$$u_1''(t) + (1 + (u_1'(t))^2)^{3/2} = 0, \quad t \in (0, 2),$$

$$u_1(0) = 0, \quad u_1(2) = 0,$$

has a solution the upper half of the circle

$$(t - 1)^2 + u_1^2 = 1,$$

having infinite derivatives at $t = 0$ and $t = 2$.

Now we give the general definition of Nagumo condition relative to upper and lower solutions.

Definition 1.3.1. (see, [27]) Suppose $f_1 : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ and $\alpha_1, \beta_1 : [0, 1] \to \mathbb{R}$ are continuous functions provided $\alpha_1(t) \le \beta_1(t)$ on $[0, 1]$, and suppose that there is a function $g : [0, \infty) \to (0, \infty)$, such that

$$|f_1(t, u_1, u_1')| \le g(|u_1'|), \tag{1.3.8}$$

for all $t \in [0,1]$, $\alpha_1(t) \leq u_1 \leq \beta_1(t)$, $u_1' \in \mathbb{R}$ with

$$\int_M^\infty \frac{sds}{g(s)} > \max_{0 \leq t \leq 1} \beta_1(t) - \min_{0 \leq t \leq 1} \alpha_1(t), \tag{1.3.9}$$

where

$$M := \max\{|\beta_1(1) - \alpha_1(0)|, |\alpha_1(1) - \beta_1(0)|\}, \tag{1.3.10}$$

where function g is known as Nagumo function .

To observe that Nagumo condition yields boundness to the derivative u_1' of the solution u_1 of (1.3.1), the following theorem is required [27, 102].

Theorem 1.3.1. *Suppose that $f_1 : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies a Nagumo condition on $[0,1]$ with respect to α_1, β_1 with $\alpha_1(t) \leq \beta_1(t)$. Assume $u_1 \in C^2([0,1], \mathbb{R})$ is a solution of (1.3.1) satisfying $\alpha_1(t) \leq u_1(t) \leq \beta_1(t)$, for all $t \in [0,1]$. Then for any $u_1 \in C^2([0,1], \mathbb{R})$, there exists a positive constant M_1 that depends on α_1, β_1 and Nagumo function g only, such that*

$$|u_1'(t)| \leq M_1, \quad t \in [0,1]. \tag{1.3.11}$$

Proof. Choose $M_1 > M$ due to (1.3.9) so that

$$\int_M^{M_1} \frac{sds}{g(s)} > \max_{0 \leq t \leq 1} \beta_1(t) - \min_{0 \leq t \leq 1} \alpha_1(t). \tag{1.3.12}$$

Let u_1 be the solution of $u_1'' = f_1(t, u_1, u_1')$ on $[0,1]$ satisfying $\alpha_1(t) \leq u_1(t) \leq \beta_1(t)$. Then according to mean value theorem there exists $s_0 \in [0,1]$, such that

$$|u_1'(s_0)| = |u_1(1) - u_1(0)| \leq M.$$

Suppose on contrary $|u_1'(t)| > M_1$. Then there exists $[s_1, s_2] \subset [0,1]$ such that the following cases hold:

1. $u_1'(s_1) = M, u_1'(s_2) = M_1$, and $M < u_1'(t) < M_1$, $t \in (s_1, s_2)$,

2. $u_1'(s_1) = M_1, u_1'(s_2) = M$, and $M < u_1'(t) < M_1$, $t \in (s_1, s_2)$,

3. $u_1'(s_1) = -M, u_1'(s_2) = -M_1$, and $-M_1 < u_1'(t) < -M$, $t \in (s_1, s_2)$,

4. $u_1'(s_1) = -M_1, u_1'(s_2) = -M$, and $-M_1 < u_1'(t) < -M$, $t \in (s_1, s_2)$.

Firstly, we consider case(1) on $[s_1, s_2]$ and in the light of (1.3.8), we can write

$$|u_1''(t)| = |f_1(t, u_1, u_1')| \le g(|u_1'(t)|),$$

$$|u_1''(t)||u_1'(t) = |f_1(t, u_1, u_1')| \le g(|u_1'(t)|)u_1'(t),$$

$$\frac{|u_1''(t)||u_1'(t)}{g(|u_1'(t)|)} \le u_1'(t),$$

$$\frac{u_1''(t)u_1'(t)}{g(u_1'(t))} \le u_1'(t),$$

$$\int_{s_1}^{s_2} \left(\frac{u_1''(t)u_1'(t)}{g(u_1'(t))} \right) dt \le \int_{s_1}^{s_2} u_1'(t)dt,$$

$$\int_{s_1}^{s_2} \left(\frac{u_1''(t)u_1'(t)}{g(u_1'(t))} \right) dt \le u_1(s_2) - u_1(s_1),$$

which further implies that

$$\int_{u_1'(s_1)}^{u_1'(s_2)} \frac{sds}{g(s)} \le u_1(s_2) - u_1(s_1) \le \max_{0 \le t \le 1} \beta_1(t) - \min_{0 \le t \le 1} \alpha_1(t),$$

it further follows that

$$\int_{M}^{M_1} \frac{sds}{g(s)} \le u_1(s_2) - u_1(s_1) \le \max_{0 \le t \le 1} \beta_1(t) - \min_{0 \le t \le 1} \alpha_1(t),$$

which contradicts to (1.3.12).

Now consider case (2) and again with the help of (1.3.8), it can be written as

$$\frac{u_1''(t)u_1'(t)}{g(u_1'(t))} \le u_1'(t), \quad t \in [s_1, s_2]$$

$$\int_{s_1}^{s_2} \left(\frac{u_1''(t)u_1'(t)}{g(u_1'(t))} \right) dt \le \int_{s_1}^{s_2} u_1'(t)dt,$$

$$\int_{s_1}^{s_2} \left(\frac{u_1''(t)u_1'(t)}{g(u_1'(t))} \right) dt \le u_1(s_2) - u_1(s_1),$$

which further implies that

$$\int_{u_1'(s_1)}^{u_1'(s_2)} \frac{sds}{g(s)} \le u_1(s_2) - u_1(s_1) \le \max_{0 \le t \le 1} \beta_1(t) - \min_{0 \le t \le 1} \alpha_1(t),$$

it further follows that

$$\int_{M_1}^{M} \frac{sds}{g(s)} \le u_1(s_2) - u_1(s_1) \le \max_{0 \le t \le 1} \beta_1(t) - \min_{0 \le t \le 1} \alpha_1(t),$$

which again contradicts to (1.3.12).

Now, consider case (4), and again using (1.3.8), we have

$$|u_1''(t)| = |f_1(t, u_1, u_1')| \leq g(|u_1'(t)|), \quad t \in [s_1, s_2],$$

$$|u_1''(t)||u_1'(t) = |f_1(t, u_1, u_1')| \geq g(|u_1'(t)|)u_1'(t),$$

$$\frac{|u_1''(t)||u_1'(t)}{g(|u_1'(t)|)} \geq u_1'(t),$$

$$\frac{u_1''(t)u_1'(t)}{g(-u_1'(t))} \geq u_1'(t),$$

it further implies that

$$-\frac{u_1''(t)u_1'(t)}{g(-u_1'(t))} \leq -u_1'(t),$$

$$\int_{s_1}^{s_2} \left(-\frac{u_1''(t)u_1'(t)}{g(-u_1'(t))} \right) \leq u_1(s_1) - u_1(s_2),$$

which further leads to

$$\int_{-u_1'(s_1)}^{-u_1'(s_2)} \frac{-sds}{g(s)} = \int_{M_1}^{M} \frac{-sds}{g(s)} \leq u_1(s_1) - u_1(s_2),$$

Finally, we have

$$\int_{M}^{M_1} \frac{sds}{g(s)} \leq u_1(s_1) - u_1(s_2) \leq \max_{0 \leq t \leq 1} \beta_1(t) - \min_{0 \leq t \leq 1} \alpha_1(t).$$

Again there is a contradiction. Case (3) can be dealt is a similar way. Hence, it can be concluded that (1.3.11) is valid. $\qquad \square$

1.4   Fixed Point Theory

Fixed point theory (FPT) plays an important role in establishing the classical results of modern mathematics. It is a gorgeous combination of geometry, analysis, and topology. Theorems relating to the properties and existence of fixed points are identified with the name of fixed point theorems. Fixed point theorems has numerous applications in almost every field of science, the names of few are, chemistry, physics, engineering, economics, game theory, biology, medical science, and much more, see for example [30, 32, 79, 134, 172, 190]. In this Section, the short note on the development of FPT along with some fixed point results is presented.

1.4.1 Development of Fixed Point Theory

Assume f_1 ba a mapping on a nonempty set X such that $f_1 : X \to X$, if $f_1(x) = x$, then the point $x \in X$ is known to be a fixed point (FP) of the map f_1. On the other hand, if f_1 ba a multi-valued map, i.e., $f_1 : X \to 2^X$, and $x \in f_1(x)$ then the point $x \in X$ is known to be a FP of the map f_1. The main advantage of the FPT is the ability to transform the wide class of equations occurring in a variety of physical formulations into a fixed point inclusions or equations. In 1837, Liouville [124] gave birth to the idea of FPT by developing the theory of successive approximations used to investigate the existence of solutions of DEs. The other notable contribution regarding the existence of solutions of DEs applying FPT was due to the work of Picard [151] in 1890. But the major breakthrough in the field of nonlinear analysis was become possible in twentieth century due to the pioneering work of well renowned mathematician Stefan Banch [33]. He gave the very important result, known as Banach fixed point theorem, which states that *"a contraction mapping on a complete metric space has a unique fixed point"*. In 1912, Brouwer [34] presented a very strong result that led the foundations of classical FPT, that states that *"a continuous map on a closed unit ball in $\mathbb{R}^n$ has a fixed point"*. In 1930, this idea was extended in Schauders fixed point theorem, that states that *"a continuous map on a convex compact subspace of a Banach space has a fixed point"* [165]. Afterward a number of researchers, for example (Tychonoff [173], Kakutani [108], Lefschetz [125], Tarski [174], Edelstein [63], Kannan [109], Chatterjea [43]) enriched the FPT by extending these distinguished results. Banach emphasized the the map f_1 should be continuous on space X but in 1968, Kannan [109] presented the fixed point theorem for operators that need not be continuous.

1.4.2 Some basic definitions and known results

Some known results form functional analysis and basic definitions is given in this section. For detailed study the reader can consult [54, 89].

Definition 1.4.1. [Compact] A subset X of a Banach space $\mathfrak{B}$ is said to be compact if every sequence in X has a subsequence that converges to an element contained in X.

Definition 1.4.2. [Relatively Compact] A subset X of a Banach space $\mathfrak{B}$ is said to be relatively compact if and only if its closure $\overline{X}$ is compact.

Definition 1.4.3. [Compact map] Assume X be a subset of a Banach space $\mathfrak{B}$ and $f_1 : X \to \mathfrak{B}$. If f_1 maps every bounded subset of X into a relatively compact subset of $\mathfrak{B}$ then $f_1 : X \to \mathfrak{B}$ is compact.

If $f_1 : X \to \mathfrak{B}$ is continuous and compact then it is completely continuous map.

Theorem 1.4.1. [Arzelà–Ascoli Theorem] *If $\mho$ be the subset of $\mathbb{R}^n$, $C(\overline{\mho})$ represents the Banach space of continuous functions and X be a subset of $C(\overline{\mho})$ then X is said to be relatively compact if and only if it is equibounded and equicontinuous.*

1.5   Fractional Calculus

In this section, we discuss some preliminary concepts of fractional calculus, some results from matrix theory, and some results related to approximation theory. We give the general idea of fractional integrals and derivatives with the aid of Euler Gamma function.

1.5.1   Fractional order differentiation and integration

The generalization of factorial function to Gamma function leads to the concept of derivatives of fractional order.

Definition 1.5.1. [148] In 1729, Euler developed the idea of Gamma function while working on the interpolation problem for factorial function. It is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \cdots\}.$$

The Cauchy's integral [159] formula for n–fold integral assists us to define fractional order differentiation and integration. That is

$$\begin{aligned}
\mathcal{I}^m g(s) &= \int_b^s \int_b^{s_{m-1}} \cdots \int_b^{s_1} g(s_0) ds_0 \cdots ds_{m-2} ds_{m-1} \\
&= \frac{1}{(m-1)!} \int_b^s (s-t)^{m-1} g(t) dt, \quad n \in \mathbb{N},
\end{aligned} \tag{1.5.1}$$

where $g \in \mathcal{L}_2[a,b], a, b \in \mathbb{R}$. If integral on right hand side in (1.5.1) converges then Euler's Gamma function permits us to replace m in (1.5.1) with an arbitrary $\alpha \in \mathbb{R}$. The Riemann-Liouville fractional-integral of order α can be defined with the help of this natural extension as

Definition 1.5.2. [101, 148] The integral of fractional order $\alpha \in \mathbb{R}_+$ of a function $\varphi \in (L^1[[c,d],\mathbb{R}])$ with $[c,d] \subset \mathbb{R}$ in terms of Riemann-Liouville approach is defined as,

$$\mathcal{I}_c^\alpha \varphi(s) = \frac{1}{\Gamma(\alpha)} \int_c^s (s-t)^{\alpha-1} \varphi(t) dt, \tag{1.5.2}$$

provided that the integral on right hand side exists.

(1.5.2) implies Riemann version for finite lower limit of integral and it implies Liouville version of fractional–integral for infinite lower limit of integral. The relation 1.5.2 is known as Riemann-Liouville fractional integral for the case when lower limit of integral is zero. In 1823, Abel solved his celebrated integral equation using fractional integral operator $\mathcal{I}^{\frac{1}{2}}$. At later stage he extended the idea for fractional order α lies in 0 and 1, i.e. $0 < \alpha < 1$.

Now it is the time to develop the idea for derivatives of fractional order and their important properties that are frequently used in the approximation theory.

If the notation $\mathcal{D}^n$ is used to represent the n–th order derivative then fundamental theorem for calculus of integer order can be written as

$$\mathcal{D}\mathcal{I}_a g(t) = g(t), \tag{1.5.3}$$

As an outcome of the repeated application of (1.5.3), we can write

$$\mathcal{D}^n \mathcal{I}_a^n g(t) = g(t). \tag{1.5.4}$$

Applying $\mathcal{D}^n$ on (1.5.4) and replacing n by $m-n$ with $n < m$, we can write

$$\mathcal{D}^n g(t) = \mathcal{D}^n \mathcal{D}^{m-n} \mathcal{I}_a^{m-n} g(t) = \mathcal{D}^m \mathcal{I}_a^{m-n} g(t). \tag{1.5.5}$$

Now we are in a position to define the following definition because (1.5.5) is still justified and workable if n is replaced by α.

Definition 1.5.3. [159] If $\alpha \in \mathbb{R}$, $m = \lceil \alpha \rceil$, and $g \in C[0,1]$, then Riemann-Liouville derivative of fractional order α can be defined as

$$\mathcal{D}^\alpha g = \mathcal{D}_a^m \mathcal{I}_a^{\alpha-m} g. \tag{1.5.6}$$

By taking the left inverse of Riemann-Liouville integral of fractional order we get Riemann-Liouville derivative of fractional order but in general the converse does not hold with exception

of some classified class of functions. Now we discuss the following lemma and some limitations of Riemann-Liouville differential operator of fractional order.

Lemma 1.5.1. *[148] If $\alpha \in (n-1, n)$, and $m \in \mathbb{N}$, then we have the following result*

$$\mathcal{I}_a^\alpha \mathcal{D}_a^\alpha g(t) = g(t) - \sum_{j=1}^{m} \frac{(t-a)^{\alpha-j}}{\Gamma(\alpha-j+1)} \mathcal{D}_a^{\alpha-j} g(a). \tag{1.5.7}$$

The development of the differential operator of fractional order by Riemann-Liouville played some meaningful role in developing the theory of differentiation and integration of arbitrary order but his developed theory have some flaws because it does not address properly to the real world physical models. It is to be observed that the Riemann-Liouville derivative of a constant is never zero, it is zero only for the case when the lower limit of the integral is infinite. But in several physical models the requirement of the limit is finite. It is to be also observed that while adopting Riemann-Liouville approach for differential equations of fractional order subject to initial conditions directs us to the initial conditions having derivative of fractional order at lower limit. These kind of problems can be solved successfully using some valid numerical techniques. But until now, we do not have any well-known physical interpretation to initial conditions prepared by using fractional order derivatives. So, Riemann-Liouville approach do not properly suit to handle the models based on real phenomenon. In 1967, Caputo presented another definition of derivative of fractional order to minimize the problems occurred in Riemann-Liouville approach. Later on in 1969, Caputo used his developed approach in viscoelasticity theory. Now it is the time to give the formal definition of Caputo derivative of fractional order.

Definition 1.5.4. [159] The Caputo left derivative of fractional order α for a function $\varphi(s) \in C^m[b, d]$ can be expressed as

$$^c\mathcal{D}_b^\alpha \varphi(s) = \frac{1}{\Gamma(m-\alpha)} \int_b^t \frac{\varphi^{(n)}(s)}{(t-s)^{\alpha+1-m}} ds, \ m-1 \leq \alpha < m, \ m \in \mathbb{N}, \tag{1.5.8}$$

provided the right side is point wise defined on (b, ∞). Moreover, if α is not an integer then $m = [\alpha] + 1$, and if α is an integer then $n = \alpha$.

By the use of Equation 1.5.2 and Equation 1.5.8, it can be easily observed that

$$^c\mathcal{D}_0^\alpha y^l = \frac{\Gamma(1+l)}{\Gamma(1+l-\alpha)} y^{l-\alpha}, \qquad \mathcal{I}_0^\alpha y^l = \frac{\Gamma(1+l)}{\Gamma(1+l+\alpha)} y^{l+\alpha},$$

and $^c\mathcal{D}_0^\alpha Q = 0$, if D is constant. \tag{1.5.9}

Now we define a lemma which shows that the left inverse of Riemann-Liouville integral of fractional order gives us a Caputo derivative of fractional order.

Lemma 1.5.2. *[159] Assume g be a continues function and $\alpha \in \mathbb{R}$, then*

$$^{c}\mathcal{D}_{a}^{\alpha}\mathcal{I}_{a}^{\alpha}g(t) = g(t). \tag{1.5.10}$$

In general the converse of Lemma 1.5.3 does not hold except for a restricted class of functions. For this we have the following result.

Lemma 1.5.3. *[159] Assume $g \in AC^{m}[a,b]$, $\alpha > 0$, and $m = \lceil \alpha \rceil$ then*

$$\mathcal{I}_{a}^{\alpha}[^{c}\mathcal{D}_{a}^{\alpha}g(t)] = g(t) - \sum_{i=0}^{m-1} \frac{\mathcal{D}^{(i)}g(a)}{i!}(t-a)^{i}.$$

1.6 Approximation Theory

In 1853, a well–known Russian mathematician, Chebyshev gave birth to the approximation theory, while investigating the problem of *linkages*, devices that translate the linear motion of a steam engine into the circular motion of a wheel, considered the following problem:

If f_1 be a continuous function on $[a,b]$ then is it possible to construct a polynomial $p_1(x) = \sum_{i=0}^{n} c_i x^i$, of degree at most n, such that the error $(\max_{a \le x \le b} |f_1(x) - p_1(x)|)$ is minimized?

The construction and choice of $p_1(x)$ is very important to get the accuracy in numerical results. Currently, many scientists are using the orthogonal polynomials to develop the approximation theory and efficient numerical schemes for integer and fractional order differential equations. The names of few of among are, Hermite polynomials, Laguerre polynomials, Jacobi polynomials, and Legendre polynomials. In this thesis we are using the generalized two parametric shifted Jacobi polynomials (JPs), which is defined as

1.6.1 Shifted Jacobi Polynomials [75, 163]

A well reputed scientist Carl Gustav Jacob Jacobi (1804–1851) defined the Jacobi polynomials which are actually the solutions of homogeneous differential equations of the second order of the form

$$(1 - z^2)u_1''(x) + (\alpha - \beta - (\alpha + \beta + 2)x)u_1'(x) + n(n + \beta + \alpha + 1)u_1(x) = 0. \tag{1.6.1}$$

If we choose $\beta, \alpha > -1$, and $n \in \mathbb{N}$ then a polynomial of order n is solution of (1.6.1). It is defined as

$$Q_n^{(\beta,\alpha)}(x) = \frac{\Gamma(\beta + n + 1)}{n!\Gamma(\beta + \alpha + n + 1)} \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) \frac{\Gamma(n + m + \beta + \alpha + 1)}{\Gamma(\beta + n + 1)} \left(\frac{x-1}{2} \right)^m. \qquad (1.6.2)$$

It is to be noted that JPs are orthogonal on the interval $[-1, 1]$ with respect to the weight function $(1-x)^\beta (1+x)^\alpha$. Under the transformation $x = \frac{2z}{\Delta} - 1$, we can obtained the two–parametric shifted JPs, defined as

$$Q_{\Delta,i}^{(\beta,\alpha)}(z) = \sum_{k=0}^{i} \frac{(-1)^{i-k}\Gamma(i + \alpha + 1)\Gamma(i + k + \beta + \alpha + 1)}{\Gamma(k + \alpha + 1)\Gamma(i + \beta + \alpha + 1)(i - k)!k!\eta^k} z^k, \; i = 0, 1, 2, 3, ... \qquad (1.6.3)$$

In the light of (1.6.3), the following relations are obtained [75]

$$Q_{\Delta,i}^{(\beta,\alpha)}(0) = (-1)^i \frac{\Gamma(i + \alpha + 1)}{\Gamma(i + 1)\Gamma(\alpha + 1)},$$
$$Q_{\Delta,i}^{(\beta,\alpha)}(\Delta) = \frac{\Gamma(i + \beta + 1)}{\Gamma(i + 1)\Gamma(\beta + 1)}, \qquad (1.6.4)$$

and

$$max_{z \in [0,\Delta]}|Q_{\Delta,i}^{(\beta,\alpha)}(z)| \leq \hat{D}(i, \zeta), \qquad (1.6.5)$$

where $\hat{D}(i, \zeta) = \frac{\Gamma(i+\zeta+1)}{\Gamma(i+1)\Gamma(\zeta+1)}$ and $\zeta = max(\sigma, \gamma)$.

The orthogonality condition of JPs on $[0, \Delta]$ is as under

$$\int_0^{\Delta} Q_{\Delta,i}^{(\beta,\alpha)}(z)Q_{\Delta,j}^{(\beta,\alpha)}(z)R_{\Delta}^{(\beta,\alpha)}(z)dz = W_{\Delta,j}^{(\beta,\alpha)}\delta_{(i,j)}, \qquad (1.6.6)$$

where the weight function $R_{\Delta}^{(\beta,\alpha)}(z)$ has the following form

$$R_{\Delta}^{(\beta,\alpha)}(z) = (\Delta - z)^\beta z^\alpha, \qquad (1.6.7)$$

and

$$W_{\Delta,j}^{(\beta,\alpha)} = \frac{\Delta^{\beta+\alpha+1}\Gamma(j + \beta + 1)\Gamma(j + \alpha + 1)}{(2j + \beta + \alpha + 1)\Gamma(j + 1)\Gamma(j + \beta + \alpha + 1)}. \qquad (1.6.8)$$

Due to orthogonal behaviour of JPs, any function $f_1(z) \in C[0, \Delta]$ can be expressed in terms of a linear combination of shifted JPs, as

$$f_1(z) = \sum_{b=0}^{\infty} c_b Q_{\Delta,b}^{(\beta,\alpha)}(z). \qquad (1.6.9)$$

Due to the practicality of the numerical problems we are interested in finding the truncated sum, therefore we can write (1.6.9) in terms of truncated sum, as

$$f_1(z) \simeq \sum_{b=0}^{M} c_b Q_{\Delta,b}^{(\beta,\alpha)}(z). \tag{1.6.10}$$

The obtained numerical results approaches to the exact results as $N \to \infty$. Therefore with the help of (1.6.6), (1.6.7), and (1.6.8), the coefficients $c(b)$ can be easily determined. In vector notation (1.6.10) has the following form

$$f_1(z) \simeq \mathbf{B}_M^T \mathbf{\Psi}_M(z), \tag{1.6.11}$$

Where $M = m + 1$ is actually a size of vectors and we use it as a scale level to develop the examples of the proposed numerical schemes. The coefficient vector is $\mathbf{B}_M^T$ and finally $\mathbf{\Psi}_M(x)$ represents the function vector. The reader can consult [75, 163] for the detail study of JPs.

1.6.2   Error Analysis

Practically, It is very important to take the finite terms while dealing to the series representation of a sufficiently smooth function $f_1 \in C[a,b]$. Assume $\prod_M(z)$ denotes the space of M terms shifted JPs and the best approximation of $f_1(z)$ is $f_{(M)}(z)$ in the given space $\prod_{(M)}(z)$. Then it leads us to the following relation

$$\|f_1(z) - f_{(M)}(z)\|_2 \le \|f_1(z) - E_{(M)}(z)\|_2, \tag{1.6.12}$$

where $E_{(M)}(z)$ is an arbitrary polynomial having degree $\le M$. Then following the same procedure as mentioned in [76], the following relation can easily be obtained representing the error of approximation

$$\|f_1(z) - E_{(M)}(z)\|_2 \le \frac{K_1}{M^{M+1}}, \tag{1.6.13}$$

where

$$K_1 = \frac{1}{4} \max_{z \in [a,b]} |\frac{\partial^{M+1}}{\partial z^{M+1}} f_1(z)|. \tag{1.6.14}$$

But in our case K_1 will be calculated on the domain $[0, \Delta]$ instead of $[a,b]$. Hence (1.6.14) becomes

$$K_1 = \frac{1}{4} \max_{z \in [0,\Delta]} |\frac{\partial^{M+1}}{\partial z^{M+1}} f_1(z)|.$$

1.6.3   Spectral Accuracy

The decay of the expansion coefficients and spectral accuracy can be investigated with the help of the following lemma [31, 88].

Lemma 1.6.1. *If $\prod_N(z)$ is taken as the space of orthogonal–shifted JPs of N terms then for any $f_1(z) = \sum_{j=0}^M b_l Q_{\tau,j}^{(\beta,\alpha)}(z) \in \prod_N(z)$, we have the following rusult;*

$$| b_j | \simeq \frac{K}{(\rho_l)^M} \parallel f_{1_{(M)}} \parallel \ and \parallel f_1(z) - \sum_{l=0}^M d_l Q_{\tau,l}^{(\beta,\alpha)}(z) \parallel^2 = \sum_{l=M}^\infty \rho_l b_l^2, \qquad (1.6.15)$$

where $\rho_l = l(l + \beta + \alpha + 1)$ and $d_l = \frac{1}{R_{\tau,j}^{(\beta,\alpha)}} \int_0^\tau f_1(z) Q_{\tau,l}^{(\beta,\alpha)}(z) R_\tau^{(\alpha,\beta)}(z) dz$. Moreover, K is dealt as a constant and M can be chosen in a way such that $f_{1_{(2M)}} \in \prod_N(z)$. Also we have the equality

$$f_{1_{(M)}} = \frac{1}{R_\tau^{(\beta,\alpha)}(z)} L f_{1_{(M-1)}}(z) = \left(\frac{L}{R_\tau^{(\beta,\alpha)}(z)} \right)^M f_1(z),$$

where L is the Sturm-Liouville operator and $f_{1_{(0)}} = f_1(z)$.

It is to be noted that in the presence of some specific conditions JPs are well suited to expand an arbitrary function defined on a finite interval. Also the Jacobi spectral expansion and the convergence of any function defined on a finite interval depends upon the power decay of Jacobi spectral expansion coefficients. Moreover it is evident form Lemma 1.6.1 that for any function $f_1(z) \in C^\infty[0, 1]$, it is possible to recover the spectral decay using expansion coefficients. We mean to say that $|d_l|$ decays faster than any algebraic order of ρ_l. It is worth to mention that this result does not depend on the BCs. For detail study on the analytical results of JPs we refer the reader to [31, 88].

1.7   Matrix Theory

Our developed numerical scheme is based on the operational matrices of integrals and derivatives of fractional order that are used to transform the considered FOPDEs into a system of easily solvable algebraic equations. In this section we are interested in defining these special type of matrix equations.

1.7.1 Sylvester Equation

Let A be a square matric of order m with real entries and B be a square matrix of order n with real entries then Sylvester equation can be written in the following form as

$$AX + XB = C. \tag{1.7.1}$$

In the light of Sylvester theorem [123, 164] for every matrix C with real entries of order $m \times n$, (1.7.1) has a unique solution X of order $m \times n$ if and only if the intersection of $\alpha(A)$ and $\alpha(B)$ generates an empty set, where $\alpha(A)$ and $\alpha(B)$ represents the spectrums of the matrices A and B respectively. Numerous applications regarding to Sylvester equations are available in the literature, the names of few of them are, control theory, model reduction, filtering, signal processing, image restoration, decoupling techniques for ordinary and partial differential equations, implementation of implicit numerical methods for ordinary differential equations, block-diagonalization of matrices, and much more, see for emaple [42, 53, 61, 62, 77, 164]. Some direct methods like Stewart method [29], and Hessenberg–Schur method [61, 77] are used to solve (1.7.1). Under these methods the coefficient matrices in (1.7.1) are transformed into Schur or Hessenberg form and then the corresponding system of linear equations are solved applying back-substitution technique. But these methods are still unable to provide an explicit formula for the solutions.

1.7.2 Vandermonde matrix

Vandermonde matrix is known by the name of a famous researcher Alexandre-Theoophile Vandermonde, is a matrix of scalars $\kappa_1, \kappa_2, \cdots, \kappa_n$ and defined as

$$H(\kappa_1, \kappa_2, \cdots, \kappa_n) = \begin{pmatrix} \kappa_1 & \kappa_1^2 & \cdots & \kappa_1^n \\ \kappa_2 & \kappa_2^2 & \cdots & \kappa_2^n \\ \vdots & \vdots & \ddots & \vdots \\ \kappa_n & \kappa_n^n & \cdots & \kappa_n^n \end{pmatrix}, \tag{1.7.2}$$

where $\kappa_1 < \kappa_2 < \cdots \kappa_n$. These matrices have very important role in approximation problems, like, interpolation, moment and least square problems. Its applications can also be observed in digital signal processing while computing discrete fourier transform and inverse discrete fourier transform

[69, 78, 106]. The following Lemma is used to calculate the inverse of a Vandermonde matrix and will be used for our later work to calculate the inverse of operational matrices developed using JPs.

Lemma 1.7.1. *The inverse of a matrix* (1.7.2) *exists and is defined as*

$$H^{-1} = [c_{(s,r)}],$$

where

$$c_{(s,r)} = \frac{1}{\kappa_s}(-1)^l \left(\frac{\sum\limits_{\substack{0 \le n_1 < \dots < n_{n-k} \le m \\ n_1,\dots,n_{q-l} \ne s}} \kappa_{n_1} \cdots \kappa_{n_{q-l}}}{\prod\limits_{\substack{0 \le n \le m \\ n \ne s}} (\kappa_n - \kappa_s)} \right). \qquad (1.7.3)$$

For the detail study of the above Lemma, the reader can consult [107].

Chapter 2

First-Order Nonlinear Coupled Systems With Nonlinear Coupled Boundary Conditions

The aim of this chapter is to investigate the existence of solutions of nonlinear first-order ordinary coupled systems with nonlinear CBCs using LUSs approach in the presence of the existence of a pair of coupled lower solution (α_1, α_2) and upper solution (β_1, β_2) such that $(\alpha_1, \alpha_2) \preceq (\beta_1, \beta_2)$ on $[0,1] \times [0,1]$. We also discuss the existence of solutions when lower and upper solutions are in reverse order, that is, $(\alpha_1, \alpha_2) \succeq (\beta_1, \beta_2)$ on $[0,1] \times [0,1]$.

The applications of nonlinear BVPs can be observed in almost every field of science, including population dynamics, mechanics, biotechnology, optimal control, physics, ecology, harvesting, and many more [139, 191, 192].

While dealing with nonlinear ordinary differential systems(ODSs) mostly authors only focus their attention on the differential systems with uncoupled BCs [18, 44, 91]. However, very rare research work is available where the differential systems are coupled not only in the differential systems but also through the BCs [16, 180]. We deal to the latter case.

In this chapter, we deal to the existence of solutions of the first-order nonlinear coupled systems of the type

$$
\begin{aligned}
u_1^{'}(t) &= f_1(t, u_2(t)), \quad t \in [0,1], \\
u_2^{'}(t) &= f_2(t, u_1(t)), \quad t \in [0,1],
\end{aligned}
\tag{2.0.1}
$$

subject to nonlinear CBC

$$\phi\left(u_1(0), u_2(0), u_1(1), u_2(1)\right) = (0,0), \tag{2.0.2}$$

where the nonlinear functions $f_1, f_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $\phi : \mathbb{R}^4 \to \mathbb{R}^2$ are continuous.

The very interesting and productive aspect of this chapter is the generalization of the classical existence results that had been discussed in [120, 126, 140]. We mean to say if $\phi(x, y, z, w) = (x - z, y - w)$, then (2.0.2) implies the periodic BCs. Also if $\phi(x, y, z, w) = (x + z, y + w)$, then (2.0.2) implies the anti-periodic BCs. Definitely, in order to obtain a solution satisfying some initial or BCs and lying between a lower solution and an upper solution, we need additional conditions. For example, in the periodic case it suffices that

$$\alpha_1(0) \le \alpha_1(1), \ \alpha_2(0) \le \alpha_2(1),$$
$$\beta_1(0) \ge \beta_1(1), \ \beta_2(0) \ge \beta_2(1), \tag{2.0.3}$$

and in the anti-periodic case it suffices that

$$\alpha_1(0) \le -\beta_1(1), \ \alpha_2(0) \le -\beta_2(1),$$
$$\beta_1(0) \ge -\alpha_1(1), \ \beta_2(0) \ge -\alpha_2(1). \tag{2.0.4}$$

We extend the idea of CLUSs presented in [66] for a nonlinear first-order coupled systems in Section 2.2 that allows us to get a solution in the sector $[\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$ or $[\![\beta_1, \alpha_1]\!] \times [\![\beta_2, \alpha_2]\!]$ and unifies the treatment of the classical results (2.0.3) and (2.0.4) applying some monotonicity assumptions on the arguments of the boundary function ϕ.

This chapter is organized as: In Section 2.1, the idea of LUSs is recalled. In the same section, the result presented in the Preliminary chapter in the form of the Lemma 1.2.1 is extended to calculate the inverse of the continuous operator L. In Section 2.2, the idea of CLUSs is discussed that unify the treatment of many linear and nonlinear BVPs. In Section 2.3, the existence of at-least one solution is ensured applying coupled LUSs approach. In Section 2.4, the validity of the theoretical results developed in Section 2.3 is checked with the help of an example. In Section 2.5, the existence of at-least one solution is ensured when lower and upper solutions are in reverse order. Moreover, in the same section, the validity of the theoretical results is also checked by the help of an example.

2.1 Lower and upper solutions method

In this section, we recall the concept of lower and upper solutions for the nonlinear coupled systems of the type (2.0.1) and a very useful result in the form of a lemma that is very necessary to prove the main results of the Section 2.3 and the Section 2.5.2.

Definition 2.1.1. We say that a couple of function $(\alpha_1, \alpha_2) \in C^1[0,1] \times C^1[0,1]$ is a lower solution of (2.0.1) if

$$
\begin{aligned}
\alpha_1'(t) &\leq f_1\left(t, \alpha_2(t)\right), \quad t \in [0,1], \\
\alpha_2'(t) &\leq f_2\left(t, \alpha_1(t)\right), \quad t \in [0,1].
\end{aligned}
\tag{2.1.1}
$$

In the same way, an upper solution is a function $(\beta_1, \beta_2) \in C^1[0,1] \times C^1[0,1]$ that satisfies the reversed inequalities in (2.1.1). In what follows we shall assume that $(\alpha_1, \alpha_2) \preceq (\beta_1, \beta_2)$, if $\alpha_1(t) \leq \beta_1(t)$ and $\alpha_2(t) \leq \beta_2(t)$, for all $t \in [0,1]$ or $(\alpha_1, \alpha_2) \succeq (\beta_1, \beta_2)$, if $\alpha_1(t) \geq \beta_1(t)$ and $\alpha_2(t) \geq \beta_2(t)$, for all $t \in [0,1]$.

For $u, v \in C^1[0,1]$, we define the set

$$
[\![u, v]\!] = \left\{ w \in C^1[0,1] : u(t) \leq w(t) \leq v(t), t \in [0,1] \right\}.
$$

The following lemma is important for our work.

Lemma 2.1.1. Let $L : C^1[0,1] \times C^1[0,1] \to C_0^1[0,1] \times C_0^1[0,1] \times \mathbb{R}^2$ be defined by

$$
\begin{aligned}
[L(u_1, u_2)](t) &= \left(u_1(t) - u_1(0) + \lambda \int_0^t u_1(s)\,\mathrm{d}s,\, u_2(t) - u_2(0) + \lambda \int_0^t u_2(s)\,\mathrm{d}s, \right. \\
&\qquad \left. (au_1(0) + bu_1(1), cu_2(0) + du_2(1)) \right),
\end{aligned}
\tag{2.1.2}
$$

where $\lambda, a, b, c,$ and d are real constants such that

$$
(a + be^{-\lambda})(c + de^{-\lambda}) \neq 0
$$

and

$$
C_0^1[0,1] = \left\{ w \in C^1[0,1] : w(0) = 0 \right\}.
$$

Then L^{-1} exists and is continuous and defined by

$$
\begin{aligned}
[L^{-1}(y, z, \gamma, \delta)] &= \left(e^{-\lambda t}A + y(t) - \lambda \int_0^t e^{\lambda(s-t)}y(s)\,\mathrm{d}s, \right. \\
&\qquad \left. e^{-\lambda t}B + z(t) - \lambda \int_0^t e^{\lambda(s-t)}z(s)\,\mathrm{d}s \right),
\end{aligned}
\tag{2.1.3}
$$

with

$$A = \frac{\gamma + \lambda b \int_0^1 e^{\lambda(s-1)} y(s)\, \mathrm{d}s - by(1)}{a + be^{-\lambda}},$$

$$B = \frac{\delta + \lambda d \int_0^1 e^{\lambda(s-1)} z(s)\, \mathrm{d}s - dz(1)}{c + de^{-\lambda}}.$$

Proof. Choose

$$y(t) = u_1(t) - u_1(0) + \lambda \int_0^t u_1(s)\mathrm{d}s, \tag{2.1.4}$$

$$z(t) = u_2(t) - u_2(0) + \lambda \int_0^t u_2(s)\, \mathrm{d}s, \tag{2.1.5}$$

$$\gamma = au_1(0) + bu_1(1), \tag{2.1.6}$$

and

$$\delta = cu_2(0) + du_2(1). \tag{2.1.7}$$

In the light of (2.1.4)-(2.1.7), (2.1.2) can also be written as

$$[L(u_1, u_2)](t) = (y(t), z(t), (\gamma, \delta)). \tag{2.1.8}$$

Differentiating (2.1.4) w.r.t. t, we have

$$y'(t) = u_1'(t) + \lambda u_1(t). \tag{2.1.9}$$

Multiplying (2.1.9) with integrating factor $e^{\lambda t}$, we have

$$e^{\lambda t} y'(t) = \left(u_1(t)e^{\lambda t}\right)', \tag{2.1.10}$$

then after integrating and taking the limits of integration from 0 to t, (2.1.10) becomes

$$u_1(t) = u_1(0)e^{-\lambda t} + y(t) - \lambda \int_0^t e^{\lambda(s-t)} y(s)ds, \tag{2.1.11}$$

$u_1(0)$ can easily be determined with the help of (2.1.6) as

$$\gamma = \left(a + be^{-\lambda}\right) u_1(0) + by(1) - b\lambda \int_0^1 e^{\lambda(s-1)} y(s)ds,$$

then

$$u_1(0) = \frac{\gamma + b\lambda \int_0^1 e^{\lambda(s-1)} y(s)ds - by(1)}{a + be^{-\lambda}}, \quad a + be^{-\lambda} \neq 0, \tag{2.1.12}$$

for simplicity of notation, let

$$A = \frac{\gamma + b\lambda \int_0^1 e^{\lambda(s-1)} y(s)ds - by(1)}{a + be^{-\lambda}}, \quad a + be^{-\lambda} \neq 0. \tag{2.1.13}$$

Using (2.1.13) in (2.1.11), we have

$$u_1(t) = Ae^{-\lambda t} + y(t) - \lambda \int_0^t e^{\lambda(s-t)} y(s)ds. \tag{2.1.14}$$

Similarly along the same lines, it can easily be shown that

$$u_2(t) = Be^{-\lambda t} + z(t) - \lambda \int_0^t e^{\lambda(s-t)} z(s) ds, \qquad (2.1.15)$$

with

$$B = \frac{\delta + d\lambda \int_0^1 e^{\lambda(s-1)} z(s) ds - dz(1)}{c + de^{-\lambda}}, \quad c + de^{-\lambda} \neq 0; \qquad (2.1.16)$$

(2.1.8) can also be written as

$$\left[L^{-1} \left(y(t), z(t), (\gamma, \delta) \right) \right] = (u_1(t), u_2(t)). \qquad (2.1.17)$$

Hence, (2.1.13)-(2.1.16) prove the result. □

2.2 Coupled lower and upper solutions

In this section we define the concept of CLUSs that unifies the treatment of many linear and non linear first order BVPs in the presence of some monotonicity assumptions on the arguments of the boundary function ϕ. Now we look towards the definition of CLUSs.

Definition 2.2.1. We say that a couple of functions $((\alpha_1, \alpha_2), (\beta_1, \beta_2)) \in C^1[0,1] \times C^1[0,1]$ are coupled lower and upper solutions for the problem (2.0.1) and (2.0.2) if (α_1, α_2) is a lower solution and (β_1, β_2) an upper solution for the system (2.0.1) with $(\alpha_1, \alpha_2) \preceq (\beta_1, \beta_2)$, such that

$$\phi\left(\alpha_1(0), \alpha_2(0), \alpha_1(1), \alpha_2(1)\right) \preceq (0,0) \preceq \phi\left(\beta_1(0), \beta_2(0), \beta_1(1), \beta_2(1)\right) \qquad (2.2.1)$$

and

$$\phi\left(\alpha_1(0), \alpha_2(0), \beta_1(1), \beta_2(1)\right) \preceq (0,0) \preceq \phi\left(\beta_1(0), \beta_2(0), \alpha_1(1), \alpha_2(1)\right). \qquad (2.2.2)$$

2.3 Existence of at least one solution

In this section we develop a result which ensures the existence of at least one solution of the problem (2.0.1)-(2.0.2). The statement and proof of the result is as under:

Theorem 2.3.1. *Assume that a couple of functions (α_1, α_2) and (β_1, β_2) are coupled lower and upper solutions respectively for the coupled BVPs of the type (2.0.1)-(2.0.2). In addition, suppose that the functions*

$$\phi_{(\alpha_1, \alpha_2)}(x, y) := \phi\left(\alpha_1(0), \alpha_2(0), x, y\right),$$
$$\phi_{(\beta_1, \beta_2)}(x, y) := \phi\left(\beta_1(0), \beta_2(0), x, y\right),$$

are monotone on $[\alpha_1(1), \beta_1(1)] \times [\alpha_2(1), \beta_2(1)]$, then the coupled BVP (2.0.1)-(2.0.2) has at least one solution $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$.

Proof. Let $\lambda > 0$, and consider the modified coupled BVP

$$u_1^{'}(t) + \lambda u_1(t) = F_1^*(t, u_1(t), u_2(t)), \quad t \in [0, 1],$$
$$u_2^{'}(t) + \lambda u_2(t) = F_2^*(t, u_1(t), u_2(t)), \quad t \in [0, 1],$$
$$\phi^* (u_1(0), u_2(0), u_1(1), u_2(1)) = (u_1(0), u_2(0)), \quad (2.3.1)$$

with

$$F_1^*(t, u_1(t), u_2(t)) = \begin{cases} f_1\left(t, \beta_2(t)\right) + \lambda\beta_1(t) & \text{if } (u_1(t), u_2(t)) \npreceq (\beta_1(t), \beta_2(t)), \\ f_1\left(t, u_2(t)\right) + \lambda\beta_1(t) & \text{if } \alpha_2(t) \leq u_2(t) \leq \beta_2(t), u_1(t) > \beta_1(t), \\ f_1\left(t, \alpha_2(t)\right) + \lambda\beta_1(t) & \text{if } u_2(t) < \alpha_2(t), u_1(t) > \beta_1(t), \\ f_1\left(t, \beta_2(t)\right) + \lambda u_1(t) & \text{if } u_2(t) > \beta_2(t), \alpha_1(t) \leq u_1(t) \leq \beta_1(t), \\ f_1\left(t, u_2(t)\right) + \lambda u_1(t) & \text{if } \alpha_2(t) \leq u_2(t) \leq \beta_2(t), \\ & \qquad \alpha_1(t) \leq u_1(t) \leq \beta_1(t), \\ f_1\left(t, \alpha_2(t)\right) + \lambda u_1(t) & \text{if } u_2(t) < \alpha_2(t), \alpha_1(t) \leq u_1(t) \leq \beta_1(t), \\ f_1\left(t, \beta_2(t)\right) + \lambda\alpha_1(t) & \text{if } u_2(t) > \beta_2(t), u_1(t) < \alpha_1(t), \\ f_1\left(t, u_2(t)\right) + \lambda\alpha_1(t) & \text{if } \alpha_2(t) \leq u_2(t) \leq \beta_2(t), u_1(t) < \alpha_1(t), \\ f_1\left(t, \alpha_2(t)\right) + \lambda\alpha_1(t) & \text{if } u_2(t) < \alpha_2(t), u_1(t) < \alpha_1(t), \end{cases}$$

and

$$F_2^*(t, u_1(t), u_2(t)) = \begin{cases} f_2\left(t, \beta_1(t)\right) + \lambda\beta_2(t) & \text{if } (u_1(t), u_2(t)) \npreceq (\beta_1(t), \beta_2(t)), \\ f_2\left(t, u_1(t)\right) + \lambda\beta_2(t) & \text{if } \alpha_1(t) \leq u_1(t) \leq \beta_1(t), u_2(t) > \beta_2(t), \\ f_2\left(t, \alpha_1(t)\right) + \lambda\beta_2(t) & \text{if } u_1(t) < \alpha_1(t), u_2(t) > \beta_2(t), \\ f_2\left(t, \beta_1(t)\right) + \lambda u_2(t) & \text{if } u_1(t) > \beta_1(t), \alpha_2(t) \leq u_2(t) \leq \beta_2(t), \\ f_2\left(t, u_1(t)\right) + \lambda u_2(t) & \text{if } \alpha_1(t) \leq u_1(t) \leq \beta_1(t), \\ & \qquad \alpha_2(t) \leq u_2(t) \leq \beta_2(t), \\ f_2\left(t, \alpha_1(t)\right) + \lambda u_2(t) & \text{if } u_1(t) < \alpha_1(t), \alpha_2(t) \leq u_2(t) \leq \beta_2(t), \\ f_2\left(t, \beta_1(t)\right) + \lambda\alpha_2(t) & \text{if } u_1(t) > \beta_1(t), u_2(t) < \alpha_2(t), \\ f_2\left(t, u_1(t)\right) + \lambda\alpha_2(t) & \text{if } \alpha_1(t) \leq u_1(t) \leq \beta_1(t), u_2(t) < \alpha_2(t), \\ f_2\left(t, \alpha_1(t)\right) + \lambda\alpha_2(t) & \text{if } u_1(t) < \alpha_1(t), u_2(t) < \alpha_2(t), \end{cases}$$

$$\phi^*(x, y, z, w) = p(0, (x, y)) - \phi(x, y, z, w),$$

and

$$p(t, (x, y)) = \begin{cases} (\beta_1(t), \beta_2(t)) & \text{if } (x, y) \npreceq (\beta_1(t), \beta_2(t)), \\ (x, y) & \text{if } (\alpha_1(t), \alpha_2(t)) \preceq (x, y) \preceq (\beta_1(t), \beta_2(t)), \\ (\alpha_1(t), \alpha_2(t)) & \text{if } (x, y) \nsucceq (\alpha_1(t), \alpha_2(t)). \end{cases}$$

Note that if $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$ is a solution of the coupled BVP (2.5.3), then (u_1, u_2) is a solution of the coupled BVP (2.0.1)-(2.0.2).

For the sake of simplicity we divide the proof into three steps:
We define the mappings

$$L, N : C^1[0, 1] \times C^1[0, 1] \to C_0^1[0, 1] \times C_0^1[0, 1] \times \mathbb{R}^2,$$

by

$$[L(u_1, u_2)](t) = \left(u_1(t) - u_1(0) + \lambda \int_0^t u_1(s)\, \mathrm{d}s, u_2(t) - u_2(0) + \lambda \int_0^t u_2(s)\, \mathrm{d}s, \right.$$
$$\left. (u_1(0), u_2(0)) \right)$$

and

$$[N(u_1, u_2)](t) = \left(\int_0^t F_1^*(s, u_1(s), u_2(s))\mathrm{d}s, \int_0^t F_2^*(s, u_1(s), u_2(s))\mathrm{d}s, \right.$$
$$\left. \phi^*(u_1(0), u_2(0), u_1(1), u_2(1)) \right).$$

Since $F_1^*(s, u_1(s), u_2(s))$ and $F_2^*(s, u_1(s), u_2(s))$ are bounded on $[0,1] \times \mathbb{R}^2$ and integral is a continuous function on $[0,1]$. Further ϕ^* being constant function is continuous. Therefore $[N(u_1, u_2)]$ is continuous on $[0,1]$. Further, the class $\{N(u_1, u_2) : u_1, u_2 \in C^1[0,1]\}$ is uniformly bounded and equicontinuous. Therefore in view of Arzelà-Ascoli theorem $\{N(u_1, u_2) : u_1, u_2 \in C^1[0,1]\}$ is relatively compact. Consequently N is a compact map. Moreover from Lemma 2.1.1 with $a = 1$, $b = 0$, $c = 1$, and $d = 0$, L^{-1} exists and is continuous.

On the other hand, solving (2.5.3) is equivalent to finding a fixed point of

$$L^{-1}N : C^1[0,1] \times C^1[0,1] \to C^1[0,1] \times C^1[0,1].$$

Now the Schauder fixed point theorem guarantees the existence of at least a fixed point since $L^{-1}N$ is continuous and compact.

Step 2: It remains to show that $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$.

We claim that $(u_1, u_2) \preceq (\beta_1, \beta_2)$. If $(u_1, u_2) \not\preceq (\beta_1, \beta_2)$, then $u_1 \not\preceq \beta_1$ and/or $u_2 \not\preceq \beta_2$. If $u_1 \not\preceq \beta_1$, then there exist some $r_0 \in [0,1]$, such that $u_1 - \beta_1$ attains a positive maximum at $r_0 \in [0,1]$. We shall consider three cases.

Case 1. $r_0 \in (0,1]$. Then there exists $\xi \in (0, r_0)$, such that $0 < u_1(t) - \beta_1(t) < u_1(r_0) - \beta_1(r_0)$, for all $t \in [\xi, r_0)$. This yields a contradiction, since

$$\beta_1(r_0) - \beta_1(\xi) < u_1(r_0) - u_1(\xi)$$
$$= \int_\xi^{r_0} \left(f_1(s, \beta_2(s)) - \lambda(u_1(s) - \beta_1(s)) \right) ds$$
$$< \int_\xi^{r_0} f_1(s, \beta_2(s))ds = \int_\xi^{r_0} \beta_1'(s)ds = \beta_1(r_0) - \beta_1(\xi).$$

Case 2. $r_0 = 0$ and ϕ_β is monotone nonincreasing. Then $u_1(0) - \beta_1(0) > 0$ or $u_2(0) - \beta_2(0) > 0$, and in view of (2.2.1), we have

$$(u(0), v(0)) = \phi^*(u_1(0), u_2(0), u_1(1), u_2(1))$$
$$= (\beta_1(0), \beta_2(0)) - \phi(\beta_1(0), \beta_2(0), u_1(1), u_2(1))$$
$$\preceq (\beta_1(0), \beta_2(0)) - \phi(\beta_1(0), \beta_2(0), \beta_1(1), \beta_2(1))$$
$$\preceq (\beta_1(0), \beta_2(0)), \tag{2.3.2}$$

a contradiction.

Case 3. Similarly ϕ_β is monotone nondecreasing. We shall change the inequality (2.3.2) by $(u_1(0), u_2(0)) \preceq (\beta_1(0), \beta_2(0)) - \phi(\beta_1(0), \beta_2(0), \alpha_1(1), \alpha_2(1))$ and again we get a contradiction. Consequently, $(u_1, u_2) \preceq (\beta_1, \beta_2)$, for all $t \in [0, 1]$. Similarly, we can show that $(u_1, u_2) \succeq (\alpha_1, \alpha_2)$, for all $t \in [0, 1]$.

Step 3: Now, it remains to show that (u_1, u_2) satisfies the boundary condition (2.0.2).

For this, we claim that

$$(\alpha_1(0), \alpha_2(0)) \preceq (u_1(0), u_2(0)) - \phi(u_1(0), u_2(0), u_1(1), u_2(1))) \preceq (\beta_1(0), \beta_2(0)). \qquad (2.3.3)$$

If $(u_1(0), u_2(0)) - \phi(u_1(0), u_2(0), u_1(1), u_2(1)) \not\preceq (\beta_1(0), \beta_2(0))$, then

$$\begin{aligned}
(u_1(0), u_2(0)) &= \phi^*(u_1(0), u_2(0), u_1(1), u_2(1)) \\
&= p(0, (u_1(0), u_2(0))) - \phi(u_1(0), u_2(0), u_1(1), u_2(1)) \\
&= (\beta_1(0), \beta_2(0)).
\end{aligned}$$

If $\phi_\beta(x, y)$ is monotone nonincreasing, then we have

$$\begin{aligned}
(u_1(0), u_2(0)) \quad - \quad & \phi(u_1(0), u_2(0), u_1(1), u_2(1)) \\
= \quad & (\beta_1(0), \beta_2(0)) - \phi(\beta_1(0), \beta_2(0), u_1(1), u_2(1)) \\
= \quad & (\beta_1(0), \beta_2(0)) - \phi_\beta(u_1(1), u_2(1)) \\
\preceq \quad & (\beta_1(0), \beta_2(0)) - \phi_\beta(\beta_1(1), \beta_2(1)) \\
= \quad & (\beta_1(0), \beta_2(0)) - \phi(\beta_1(0), \beta_2(0), \beta_1(1), \beta_2(1)) \\
\preceq \quad & (\beta_1(0), \beta_2(0)), \qquad\qquad\qquad\qquad\qquad (2.3.4)
\end{aligned}$$

a contradiction. Similarly if $\phi_\beta(x, y)$ is monotone nondecreasing, then we get the same contradiction. Consequently, (2.3.3) holds. Hence the system of BVPs (2.0.1)-(2.0.2) has a solution $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$. □

2.4 Application of the theoretical results

In this section, we check the applicability of the developed result 2.3.1.

Example 2.4.1. *Let*

$$\begin{aligned}
f_1(t, u_2(t)) &= -2u_2(t) + \gamma \sin(\omega t), \quad t \in [0, 1], \\
f_2(t, u_1(t)) &= -2u_1^3(t) + \gamma \cos(\omega t), \quad t \in [0, 1], \\
\phi(x, y, z, w) &= (x^3 - y^3, z^3 - w^3),
\end{aligned}$$

where γ and ω are positive integers. Let $\alpha_1(t) = -2\gamma$, $\alpha_2(t) = -\gamma$ and $\beta_1(t) = 2\gamma$, $\beta_2(t) = \gamma$.

In the light of the methodology adopted in Example 1.2.5 of the Preliminary chapter, it is evident that a couple of functions (α_1, α_2), (β_1, β_2) are lower and upper solutions respectively of the system (2.0.1). On the same fashion as mentioned in Example 1.2.5, it can be shown that a couple of functions (α_1, α_2) and (β_1, β_2) satisfy the system of inequalities (2.2.1)-(2.2.2). Hence by Theorem 2.3.1, the coupled BVPs of the type (2.0.1)-(2.0.2) has at least one solution $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$.

2.5 Lower and upper solutions in reverse order

In this section, we discuss the existence of solutions of the coupled BVPs of the type (2.0.1)-(2.0.2) when a pair of lower and upper solutions are in reverse order, i.e., $(\alpha_1, \alpha_2) \succeq (\beta_1, \beta_2)$. Let us see the definition of coupled lower and upper solutions for the case when $(\alpha_1(t), \alpha_2(t)) \succeq (\beta_1(t), \beta_2(t))$, for all $t \in [0, 1]$.

2.5.1 Coupled lower and upper solutions

Definition 2.5.1. We say that a couple of functions $((\alpha_1, \alpha_2), (\beta_1, \beta_2)) \in C^1[0, 1] \times C^1[0, 1]$ are coupled lower and upper solutions for the coupled BVP (2.0.1)–(2.0.2), if (α_1, α_2) is a lower solution and (β_1, β_2) an upper solution for the system (2.0.1) with $(\alpha_1, \alpha_2) \succeq (\beta_1, \beta_2)$, such that

$$\phi\left(\alpha_1(0), \alpha_2(0), \alpha_1(1), \alpha_2(1)\right) \preceq (0, 0) \preceq \phi\left(\beta_1(0), \beta_2(0), \beta_1(1), \beta_2(1)\right) \tag{2.5.1}$$

and

$$\phi\left(\beta_1(0), \beta_2(0), \alpha_1(1), \alpha_2(1)\right) \preceq (0, 0) \preceq \phi\left(\alpha_1(0), \alpha_2(0), \beta_1(1), \beta_2(1)\right). \tag{2.5.2}$$

2.5.2 Existence of at least one solution

Theorem 2.5.1. *Assume that a couple of functions (α_1, α_2) and (β_1, β_2) are coupled lower and upper solutions such that $(\alpha_1, \alpha_2) \succeq (\beta_1, \beta_2)$ for the coupled BVP (2.0.1)-(2.0.2). In addition, suppose that the functions*

$$\phi_{(\alpha_1, \alpha_2)}(x, y) := \phi\left(x, y, \alpha_1(1), \alpha_2(1)\right),$$
$$\phi_{(\beta_1, \beta_2)}(x, y) := \phi\left(x, y, \beta_1(1), \beta_2(1)\right),$$

are monotone in $[\beta_1(0), \alpha_1(0)] \times [\beta_2(0), \alpha_2(0)]$, then the system (2.0.1)-(2.0.2) has at least one solution $(u_1, u_2) \in [\![\beta_1, \alpha_1]\!] \times [\![\beta_2, \alpha_2]\!]$.

Proof. Let $\lambda > 0$, and consider the modified coupled BVP

$$u_1'(t) - \lambda u_1(t) = F_1^*(t, u_1(t), u_2(t)), \quad t \in [0, 1],$$
$$u_2'(t) - \lambda u_2(t) = F_2^*(t, u_1(t), u_2(t)), \quad t \in [0, 1],$$
$$\phi^*\left(u_1(0), u_2(0), u_1(1), u_2(1)\right) = (u_1(1), u_2(1)), \tag{2.5.3}$$

with

$$F_1^*(t, u_1(t), u_2(t)) = \begin{cases} f_1\left(t, \beta_2(t)\right) - \lambda \beta_1(t) & \text{if } (u_1(t), u_2(t)) \not\preceq (\beta_1(t), \beta_2(t)), \\ f_1\left(t, u_2(t)\right) - \lambda \beta_1(t) & \text{if } \alpha_2(t) \leq u_2(t) \leq \beta_2(t), u_1(t) > \beta_1(t), \\ f_1\left(t, \alpha_2(t)\right) - \lambda \beta_1(t) & \text{if } u_2(t) < \alpha_2(t), u_1(t) > \beta_1(t), \\ f_1\left(t, \beta_2(t)\right) - \lambda u_1(t) & \text{if } u_2(t) > \beta_2(t), \alpha_1(t) \leq u_1(t) \leq \beta_1(t), \\ f_1\left(t, u_2(t)\right) - \lambda u_1(t) & \text{if } \alpha_2(t) \leq u_2(t) \leq \beta_2(t), \\ & \qquad \alpha_1(t) \leq u_1(t) \leq \beta_1(t), \\ f_1\left(t, \alpha_2(t)\right) - \lambda u_1(t) & \text{if } u_2(t) < \alpha_2(t), \alpha_1(t) \leq u_1(t) \leq \beta_1(t), \\ f_1\left(t, \beta_2(t)\right) - \lambda \alpha_1(t) & \text{if } u_2(t) > \beta_2(t), u_1(t) < \alpha_1(t), \\ f_1\left(t, u_2(t)\right) - \lambda \alpha_1(t) & \text{if } \alpha_2(t) \leq u_2(t) \leq \beta_2(t), u_1(t) < \alpha_1(t), \\ f_1\left(t, \alpha_2(t)\right) - \lambda \alpha_1(t) & \text{if } u_2(t) < \alpha_2(t), u_1(t) < \alpha_1(t), \end{cases}$$

and

$$F_2^*(t, u_1(t), u_2(t)) = \begin{cases} f_2\left(t, \beta_1(t)\right) - \lambda \beta_2(t) & \text{if } (u_1(t), u_2(t)) \npreceq (\beta_1(t), \beta_2(t)), \\ f_2\left(t, u_1(t)\right) - \lambda \beta_2(t) & \text{if } \alpha_1(t) \le u_1(t) \le \beta_1(t), u_2(t) > \beta_2(t), \\ f_2\left(t, \alpha_1(t)\right) - \lambda \beta_2(t) & \text{if } u_1(t) < \alpha_1(t), u_2(t) > \beta_2(t), \\ f_2\left(t, \beta_1(t)\right) - \lambda u_2(t) & \text{if } u_1(t) > \beta_1(t), \alpha_2(t) \le u_2(t) \le \beta_2(t), \\ f_2\left(t, u_1(t)\right) - \lambda u_2(t) & \text{if } \alpha_1(t) \le u_1(t) \le \beta_1(t), \\ & \quad\quad \alpha_2(t) \le u_2(t) \le \beta_2(t), \\ f_2\left(t, \alpha_1(t)\right) - \lambda u_2(t) & \text{if } u_1(t) < \alpha_1(t), \alpha_2(t) \le u_2(t) \le \beta_2(t), \\ f_2\left(t, \beta_1(t)\right) - \lambda \alpha_2(t) & \text{if } u_1(t) > \beta_1(t), u_2(t) < \alpha_2(t), \\ f_2\left(t, u_1(t)\right) - \lambda \alpha_2(t) & \text{if } \alpha_1(t) \le u_1(t) \le \beta_1(t), u_2(t) < \alpha_2(t), \\ f_2\left(t, \alpha_1(t)\right) - \lambda \alpha_2(t) & \text{if } u_1(t) < \alpha_1(t), u_2(t) < \alpha_2(t), \end{cases}$$

$$\phi^*(x, y, z, w) = p(1, (x, y)) - \phi(x, y, z, w),$$

and

$$p(t, (x, y)) = \begin{cases} (\beta_1(t), \beta_2(t)) & \text{if } (x, y) \npreceq (\beta_1(t), \beta_2(t)), \\ (x, y) & \text{if } (\alpha_1(t), \alpha_2(t)) \preceq (x, y) \preceq (\beta_1(t), \beta_2(t)), \\ (\alpha_1(t), \alpha_2(t)) & \text{if } (x, y) \nsucceq (\alpha_1(t), \alpha_2(t)). \end{cases}$$

Now the rest of the proof is analogous to the proof of the Theorem 2.3.1 using Lemma 2.1.1 with $a = 0$, $b = 1$, $c = 0$, and $d = 1$. We left it as an exercise. $\qquad\square$

2.5.3 Application of the theoretical results

Example 2.5.1. *Let*

$$f_1(t, u_2(t)) = 4u_2(t) + \gamma \sin(\omega t), \quad t \in [0, 1],$$
$$f_2(t, u_1(t)) = 4u_1^5(t) + \gamma \cos(\omega t), \quad t \in [0, 1],$$
$$\phi(x, y, z, w) = (yw - xz, y + xz),$$

where γ and ω are positive integers. Choose $\alpha_1(t) = 3\gamma$, $\alpha_2(t) = 2\gamma$, $\beta_1(t) = -3\gamma$, $\beta_2(t) = -2\gamma$.

In the light of the methodology adopted in Example 1.2.8 of the Preliminary chapter, it is evident that a couple of functions (α_1, α_2), (β_1, β_2) are lower and upper solutions respectively in the reverse order of the system (2.0.1). On the same fashion as mentioned in Example 1.2.8, it can be shown that a couple of functions (α_1, α_2) and (β_1, β_2) satisfy the system of inequalities (2.5.1)-(2.5.2). Hence by Theorem 2.5.1, the system of BVPs (2.0.1)-(2.0.2) has at least one solution $(u_1, u_2) \in [\![\beta_1, \alpha_1]\!] \times [\![\beta_2, \alpha_2]\!]$.

Chapter 3

Second-Order Nonlinear Coupled Systems With Nonlinear Coupled Boundary Conditions

In this chapter, we extend the ideas of Chapter 2 for a second order nonlinear coupled BVPs corresponding to nonlinear CBCs in the presence of the existence of well ordered LUSs, such that $(\alpha_1, \alpha_2) \preceq (\beta_1, \beta_2)$ on $[0, 1] \times [0, 1]$.

The study of differential systems with CBCs is a very important area of analysis due to not only the theoretical aspects but also the rich applications in the existing literature; including reaction-diffusion equations, Sturm-Liouville problems, mathematical biology, chemical systems and engineering, see for example [1, 2, 5, 37, 117, 178, 188]. In 1982, Leung discussed the applications of coupled BVPs in mathematical biology by studying the following reaction-diffusion system for prey-predator interaction [117]:

$$
\begin{aligned}
\frac{\partial u_1(t, x)}{\partial t} &= \sigma_1 \Delta u_1 + u_1 \left(a + f_1(u_1, u_2) \right), \quad t \geq 0, \quad x \in \mho \subset \mathbb{R}^n, \\
\frac{\partial u_2(t, x)}{\partial t} &= \sigma_2 \Delta u_2 + u_2 \left(r + f_2(u_1, u_2) \right), \quad t \geq 0, \quad x \in \mho \subset \mathbb{R}^n,
\end{aligned}
\tag{3.0.1}
$$

subject to CBCs

$$
\frac{\partial u_1}{\partial \eta} = 0, \quad \frac{\partial u_2}{\partial \eta} - p(u_1) - q(u_2) = 0 \quad \text{on} \ \partial \mho,
\tag{3.0.2}
$$

where $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ have Hölder continuous partial derivatives up to second order in compact sets, η is a unit outward normal at $\partial \mho$ and p and q have Hölder continuous first derivatives in compact subsets of $[0, \infty)$. The functions $u_1(t, x)$ and $u_2(t, x)$ respectively represent the density

of prey and predator at $t \geq 0$ and at position $x = (x_1, x_2, ..., x_n)$. In 1978, Aronson studied the biochemical system using same type of CBCs [2]. In 2009, Cardanobile and Mugnolo studied the following parabolic system with CBCs [37]:

$$\frac{\partial u_1}{\partial t}(t, x) = \mathcal{L}u_1(t, x), \quad t \geq 0, \quad x \in \mho \in \mathbb{R}^n,$$
$$f_1 \mid \partial \mho \in \mathcal{Y}, \quad \frac{\partial f_1}{\partial u_2} \in \mathcal{Y}^\perp,$$

(3.0.3)

where $\mathcal{Y}^\perp$ is closed subspace of $L^2(\partial \Omega; W)$, the unknown function u_1 takes values in a separable Hilbert space W and $\mathcal{L}$ is an elliptic operator with operator-valued symbol.

This chapter is organized as: In Section 3.1, the method of LUSs for the second order nonlinear coupled systems is briefly discussed. In the same section, the inverse of the continuous operator L is determined by extending the result of the Chapter 2 discussed in the Section 2.1. In Section 3.1.2, the concept of CLUSs is discussed that unify the treatment of the classical existence results of many linear and nonlinear BVPs, like Dirichlet and Neumann by imposing some monotone assumptions on the arguments of the boundary functions ϕ and ψ. In Section 3.1.3, the existence of at-least one solution is ensured for the nonlinear coupled BVPs applying coupled LUSs approach. In Section 3.1.4, the validity of the theoretical results is checked by taking some examples. In Section 3.2.1, the method of LUSs is recalled for the nonlinear second-order coupled BVPs corresponding to nonlinear coupled BCs. In Section 3.2.2, the idea of CLUSs is discussed that generalizes the treatment of the classical existence results for very important BVPs, like periodic and anti-periodic by imposing some monotonicity assumptions on the arguments of the boundary functions μ and ν. In Section 3.2.3, the existence of at-least one solution of the second-order coupled BVPs is established using coupled LUSs approach. In Section 3.2.4, some illustrative examples are considered to ensure the validity of the theoretical results.

3.1 Second-order coupled system with nonlinear coupled boundary conditions (I)

In this section, we investigate the existence of solutions of the following second order ordinary nonlinear coupled BVPs of the type

$$u_1''(t) = f_1(t, u_2(t)), \quad t \in [0, 1],$$
$$u_2''(t) = f_2(t, u_1(t)), \quad t \in [0, 1],$$

(3.1.1)

with nonlinear CBCs

$$\phi\left(u_1(0), u_2(0), u_1(1), u_2(1), u_1^{'}(0), u_2^{'}(0)\right) = (0,0),$$
$$\psi\left(u_1(0), u_2(0), u_1(1), u_2(1), u_1^{'}(1), u_2^{'}(1)\right) = (0,0),$$

(3.1.2)

where $f_1, f_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $\phi, \psi : \mathbb{R}^6 \to \mathbb{R}^2$ are continuous functions. The existence results for the second order BVPs has been established by many authors in the presence of certain types of BCs, see for example [16, 86, 95, 96, 99, 181, 186, 193]. In contrast to previous studies our BCs are very much generalized because they unify the treatment of many important BVPs like Dirichlet, Neumann, periodic and anti-periodic. The detail note on the general and broad behaviour of our chosen BCs 3.1.2 and 3.2.2 has been comprehensively discussed in the introductory chapter of this thesis.

The study of differential systems becomes more interesting and some times complicated when coupling is defined among the arguments of the BCs. Systems with CBCs can be applied to interaction problems, reaction diffusion phenomena and Lotka-Volterra models, see for example [6, 128, 117].

Zhou and Xu [193] established the existence and multiplicity of positive solutions of the following nonlinear coupled system of BVPs

$$-u^{''}(t) = f(t, v(t)), \quad t \in (0,1),$$
$$-v^{''}(t) = g(t, u(t)), \quad t \in (0,1),$$
$$u(0) = v(0) = 0,$$
$$\alpha u(\eta) = u(1), \quad \alpha v(\eta) = v(1), \quad \eta \in (0,1), \text{ and } 0 < \alpha\eta < 1,$$

(3.1.3)

with uncoupled linear–multi point BCs applying the fixed point index theory in cones. Wei and Sun [184] extended this study by considering the fully nonhomogeneous uncoupled multi–point BCs of the type

$$u(0) = \alpha u(\eta), \quad u(1) = \beta u(\eta),$$
$$v(0) = \alpha v(\eta), \quad v(1) = \beta v(\eta), \quad \eta \in (0,1), \text{ and } 0 < \beta \le \alpha < 1.$$

(3.1.4)

Asif and Khan [16] extended the results for a multi point linear coupled BCs and investigated the

existence of a positive solution to the following four-point coupled BVPs by constructing a cone

$$-x''(t) = f(t, x(t), y(t)), \quad t \in (0, 1),$$

$$-y''(t) = g(t, x(t), y(t)), \quad t \in (0, 1),$$

$$x(0) = 0, \ x(1) = \alpha y(\xi),$$

$$y(0) = 0, \ y(1) = \beta x(\eta),$$

$(3.1.5)$

where $\xi, \eta \in (0, 1)$, $0 < \alpha\beta\xi\eta < 1$, $f, g : [0, 1] \times [0, \infty) \times [0, \infty) \to [0, \infty)$ are continuous and allowed to be singular at $t = 0$ and $t = 1$. On the other hand, we discuss the more generalized approach regarding the existence results for a second order nonlinear coupled systems by introducing the nonlinear coupling among the arguments of the BCs. We also claim that our considered BCs 3.1.2 are very much generalized as compared to the BCs studied in [16, 193].

3.1.1 Lower and upper solutions method

In this section, we recall the concept of LUSs and a very useful result in the form of a lemma that addresses the inverse of a compact map.

Definition 3.1.1. We say that a couple of functions $(\alpha_1, \alpha_2) \in C^2[0, 1] \times C^2[0, 1]$ is a lower solution of (3.1.1) if

$$\alpha_1''(t) \geq f_1(t, \alpha_2(t)), \quad t \in [0, 1],$$
$$\alpha_2''(t) \geq f_2(t, \alpha_1(t)), \quad t \in [0, 1]. \tag{3.1.6}$$

In the same way, an upper solution is a couple of functions $(\beta_1, \beta_2) \in C^2[0, 1] \times C^2[0, 1]$ that satisfies the reversed inequalities in (3.1.6). In what follows we will write $(\alpha_1, \alpha_2) \preceq (\beta_1, \beta_2)$, if $\alpha_1(t) \leq \beta_1(t)$ and $\alpha_2(t) \leq \beta_2(t)$, for all $t \in [0, 1]$.

For $u_1, u_2 \in C^2[0, 1]$, we define the set

$$[\![u_1, u_2]\!] = \left\{ w_1 \in C^2[0, 1] : u_1(t) \leq w_1(t) \leq u_2(t), t \in [0, 1] \right\}.$$

Let us construct the statement of the following lemma along with its proof .

Lemma 3.1.1. *Let* $L : C^2[0, 1] \times C^2[0, 1] \to C_0^2[0, 1] \times C_0^2[0, 1] \times \mathbb{R}^2 \times \mathbb{R}^2$ *be defined by*

$$[L(u_1, u_2)](t) = \left(u_1'(t) - u_1'(0) - \lambda \int_0^t u_1(s)\,ds, u_2'(t) - u_2'(0) - \lambda \int_0^t u_2(s)\,ds, \right.$$

$$\left. (au_1(0) + bu_1(1), cu_2(0) + du_2(1)), (Eu_1(0) + Fu_1(1), Gu_2(0) + Hu_2(1)) \right), (3.1.7)$$

where $\lambda, a, b, c, d, E, F, G$ and H are real constants with $\lambda > 0$, such that

$$(ad - bc)(EH - FG)\left(e^{-\sqrt{\lambda}} - e^{\sqrt{\lambda}}\right) \neq 0,$$

here

$$C_0^2[0,1] = \left\{w_1 \in C^2[0,1] : w_1(0) = 0\right\}.$$

Then L^{-1} exists and continuous, defined by

$$[L^{-1}(y,z,\gamma,\delta,\mu,\zeta)] = \left(C_1 e^{\sqrt{\lambda}t} + C_2 e^{-\sqrt{\lambda}t} + \frac{1}{2}\int_0^t e^{\sqrt{\lambda}(t-s)}y(s)\,\mathrm{d}s + \frac{1}{2}\int_0^t e^{\sqrt{\lambda}(s-t)}y(s)\,\mathrm{d}s,\right.$$
$$\left. C_3 e^{\sqrt{\lambda}t} + C_4 e^{-\sqrt{\lambda}t} + \frac{1}{2}\int_0^t e^{\sqrt{\lambda}(t-s)}z(s)\,\mathrm{d}s + \frac{1}{2}\int_0^t e^{\sqrt{\lambda}(s-t)}z(s)\,\mathrm{d}s\right),$$

(3.1.8)

with

$$
\begin{aligned}
C_1 &= \frac{1}{(ad-bc)\left(e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}\right)}\left(2\delta\left(a + be^{-\sqrt{\lambda}}\right) - d\left(a + be^{-\sqrt{\lambda}}\right)\right. \\
&\qquad \int_0^1 e^{\sqrt{\lambda}(1-s)}y(s)ds + d\left(a + be^{-\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(s-1)}y(s)ds \\
&\quad - 2\gamma\left(c + de^{-\sqrt{\lambda}}\right) + b\left(c + de^{-\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(1-s)}y(s)ds \\
&\quad - \left.b\left(c + de^{-\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(s-1)}y(s)ds\right), \\
C_2 &= \frac{1}{(ad-bc)\left(e^{-\sqrt{\lambda}} - e^{\sqrt{\lambda}}\right)}\left(2\delta\left(a + be^{\sqrt{\lambda}}\right) - d\left(a + be^{\sqrt{\lambda}}\right)\right. \\
&\qquad \int_0^1 e^{\sqrt{\lambda}(1-s)}y(s)ds + d\left(a + be^{\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(s-1)}y(s)ds \\
&\quad - 2\gamma\left(c + de^{\sqrt{\lambda}}\right) + b\left(c + de^{\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(1-s)}y(s)ds \\
&\quad - \left.b\left(c + de^{\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(s-1)}y(s)ds\right),
\end{aligned}
$$

$$
\begin{aligned}
C_3 &= \frac{1}{(EH-FG)\left(e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}\right)}\left(2\zeta\left(E + Fe^{-\sqrt{\lambda}}\right) - H\left(E + Fe^{-\sqrt{\lambda}}\right)\right. \\
&\qquad \int_0^1 e^{\sqrt{\lambda}(1-s)}z(s)ds + F\left(E + Fe^{-\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(s-1)}z(s)ds \\
&\quad - 2\mu\left(G + He^{-\sqrt{\lambda}}\right) + F\left(G + He^{-\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(1-s)}z(s)ds \\
&\quad - \left.F\left(G + He^{-\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(s-1)}z(s)ds\right),
\end{aligned}
$$

and

$$C_4 = \frac{1}{(EH - FG)\left(e^{-\sqrt{\lambda}} - e^{\sqrt{\lambda}}\right)}\left(2\zeta\left(E + Fe^{\sqrt{\lambda}}\right) - H\left(E + Fe^{\sqrt{\lambda}}\right)\right)$$

$$\int_0^1 e^{\sqrt{\lambda}(1-s)}z(s)ds + F\left(E + Fe^{\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(s-1)}z(s)ds$$

$$- \quad 2\mu\left(G + He^{\sqrt{\lambda}}\right) + F\left(G + He^{\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(1-s)}z(s)ds$$

$$- \quad F\left(G + He^{\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(s-1)}z(s)ds\Bigg).$$

Proof. Choose

$$y(t) = u_1'(t) - u_1'(0) - \lambda\int_0^t u_1(s)\,\mathrm{d}s, \tag{3.1.9}$$

$$z(t) = u_2'(t) - u_2'(0) - \lambda\int_0^t u_2(s)\,\mathrm{d}s, \tag{3.1.10}$$

$$\gamma = au_1(0) + bu_1(1), \tag{3.1.11}$$

$$\delta = cu_2(0) + du_2(1), \tag{3.1.12}$$

$$\mu = Eu_1(0) + Fu_1(1), \tag{3.1.13}$$

$$\zeta = Gu_2(0) + Hu_2(1). \tag{3.1.14}$$

With the help of (3.1.9)-(3.1.14), (3.1.7) can be written as

$$[L_1(u_1, u_2)](t) = (y(t), z(t), (\gamma, \delta), (\mu, \zeta)). \tag{3.1.15}$$

Differentiating (3.1.9) with respect to t, we have

$$y'(t) = u_1''(t) - \lambda u_1(t), \ \lambda > 0. \tag{3.1.16}$$

The general solution of (3.1.16) can be easily determined using variation of parameters technique along with integration by parts and taking limits of integration from 0 to t, we have

$$u_1(t) = C_1 e^{\sqrt{\lambda}t} + C_2 e^{-\sqrt{\lambda}t} + \frac{1}{2}\int_0^t e^{\sqrt{\lambda}(t-s)}y(s)\mathrm{d}s + \frac{1}{2}\int_0^t e^{\sqrt{\lambda}(s-t)}y(s)\mathrm{d}s. \tag{3.1.17}$$

C_1 and C_2 can be easily determined with the help of (3.1.11) and (3.1.12) as

$$\gamma = \left(a + be^{\sqrt{\lambda}}\right)C_1 + \left(a + be^{-\sqrt{\lambda}}\right)C_2 + \frac{b}{2}\left(\int_0^1 e^{\sqrt{\lambda}(1-s)}y(s)\mathrm{d}s + e^{\sqrt{\lambda}(s-1)}y(s)\mathrm{d}s\right),$$

$$\delta = \left(c + de^{\sqrt{\lambda}}\right)C_1 + \left(c + de^{-\sqrt{\lambda}}\right)C_2 + \frac{d}{2}\left(\int_0^1 e^{\sqrt{\lambda}(1-s)}y(s)\mathrm{d}s + e^{\sqrt{\lambda}(s-1)}y(s)\mathrm{d}s\right). \tag{3.1.18}$$

Solving the system of equations (3.1.18), we have

$$
\begin{aligned}
C_1 &= \frac{1}{(ad-bc)\left(e^{\sqrt{\lambda}}-e^{-\sqrt{\lambda}}\right)}\left(2\delta\left(a+be^{-\sqrt{\lambda}}\right)-d\left(a+be^{-\sqrt{\lambda}}\right)\right) \\
&\quad \int_0^1 e^{\sqrt{\lambda}(1-s)}y(s)ds + d\left(a+be^{-\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(s-1)}y(s)ds \\
&\quad - 2\gamma\left(c+de^{-\sqrt{\lambda}}\right)+b\left(c+de^{-\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(1-s)}y(s)ds \\
&\quad - b\left(c+de^{-\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(s-1)}y(s)ds\Bigg),
\end{aligned}
$$

and   (3.1.19)

$$
\begin{aligned}
C_2 &= \frac{1}{(ad-bc)\left(e^{-\sqrt{\lambda}}-e^{\sqrt{\lambda}}\right)}\left(2\delta\left(a+be^{\sqrt{\lambda}}\right)-d\left(a+be^{\sqrt{\lambda}}\right)\right) \\
&\quad \int_0^1 e^{\sqrt{\lambda}(1-s)}y(s)ds + d\left(a+be^{\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(s-1)}y(s)ds \\
&\quad - 2\gamma\left(c+de^{\sqrt{\lambda}}\right)+b\left(c+de^{\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(1-s)}y(s)ds \\
&\quad - b\left(c+de^{\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(s-1)}y(s)ds\Bigg).
\end{aligned}
$$

(3.1.20)

Similarly on the same line, it can be easily shown that

$$
u_2(t) = C_3 e^{\sqrt{\lambda}t} + C_4 e^{-\sqrt{\lambda}t} + \frac{1}{2}\int_0^t e^{\sqrt{\lambda}(t-s)}z(s)ds + \frac{1}{2}\int_0^t e^{\sqrt{\lambda}(s-t)}z(s)ds, \tag{3.1.21}
$$

with

$$
\begin{aligned}
C_3 &= \frac{1}{(EH-FG)\left(e^{\sqrt{\lambda}}-e^{-\sqrt{\lambda}}\right)}\left(2\zeta\left(E+Fe^{-\sqrt{\lambda}}\right)-H\left(E+Fe^{-\sqrt{\lambda}}\right)\right) \\
&\quad \int_0^1 e^{\sqrt{\lambda}(1-s)}z(s)ds + F\left(E+Fe^{-\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(s-1)}z(s)ds \\
&\quad - 2\mu\left(G+He^{-\sqrt{\lambda}}\right)+F\left(G+He^{-\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(1-s)}z(s)ds \\
&\quad - F\left(G+He^{-\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(s-1)}z(s)ds\Bigg),
\end{aligned}
$$

(3.1.22)

and

$$C_4 = \frac{1}{(EH-FG)\left(e^{-\sqrt{\lambda}}-e^{\sqrt{\lambda}}\right)}\left(2\zeta\left(E+Fe^{\sqrt{\lambda}}\right)-H\left(E+Fe^{\sqrt{\lambda}}\right)\right)$$
$$\int_0^1 e^{\sqrt{\lambda}(1-s)}z(s)ds + F\left(E+Fe^{\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(s-1)}z(s)ds$$
$$- \quad 2\mu\left(G+He^{\sqrt{\lambda}}\right)+F\left(G+He^{\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(1-s)}z(s)ds$$
$$- \quad F\left(G+He^{\sqrt{\lambda}}\right)\int_0^1 e^{\sqrt{\lambda}(s-1)}z(s)ds\Bigg). \tag{3.1.23}$$

(3.1.15) can also be written as

$$(u_1(t),u_2(t)) = [L^{-1}\left(y(t),z(t),(\gamma,\delta),(\mu,\zeta)\right)]. \tag{3.1.24}$$

Hence, (3.1.17)-(3.1.24) prove the required result. □

3.1.2 Coupled lower and upper solutions

In this section, we develop the definition of CLUSs that assists us to unify the treatment of many linear and non linear BVPs by imposing some monotonicity assumptions on the arguments of the boundary functions ϕ and ψ. Moreover, monotonicity assumptions on ϕ and ψ deals the classical existence criterion ((0.0.11)-(0.0.12)) for Dirichlet and Neumann BVPs as a particular case. Now, let us define the concept of CLUSs:

Definition 3.1.2. We say that a couple of functions (α_1,α_2), $(\beta_1,\beta_2) \in C^2[0,1] \times C^2[0,1]$ are coupled lower and upper solutions for the coupled BVP (3.1.1)–(3.1.2), if (α_1,α_2) is a lower solution and (β_1,β_2) is an upper solution for the system (3.1.1) with $(\alpha_1,\alpha_2) \preceq (\beta_1,\beta_2)$, such that

$$\phi(\beta_1(0),\beta_2(0),\beta_1(1),\beta_2(1),\beta_1^{'}(0),\beta_2^{'}(0)) \preceq (0,0) \preceq \phi(\alpha_1(0),\alpha_2(0),\alpha_1(1),\alpha_2(1),\alpha_1^{'}(0),\alpha_2^{'}(0)),$$
$$\phi(\beta_1(0),\beta_2(0),\alpha_1(1),\alpha_2(1),\beta_1^{'}(0),\beta_2^{'}(0)) \preceq (0,0) \preceq \phi(\alpha_1(0),\alpha_2(0),\beta_1(1),\beta_2(1),\alpha_1^{'}(0),\alpha_2^{'}(0)),$$
$$\psi(\beta_1(0),\beta_2(0),\beta_1(1),\beta_2(1),\beta_1^{'}(1),\beta_2^{'}(1)) \preceq (0,0) \preceq \psi(\alpha_1(0),\alpha_2(0),\alpha_1(1),\alpha_2(1),\alpha_1^{'}(1),\alpha_2^{'}(1)),$$
$$\psi(\alpha_1(0),\alpha_2(0),\beta_1(1),\beta_2(1),\beta_1^{'}(1),\beta_2^{'}(1)) \preceq (0,0) \preceq \psi(\beta_1(0),\beta_2(0),\alpha_1(1),\alpha_2(1),\alpha_1^{'}(1),\alpha_2^{'}(1)). \tag{3.1.25}$$

3.1.3 Existence of at least one solution

Theorem 3.1.2. *Assume that a couple of functions (α_1,α_2), (β_1,β_2) are coupled lower and upper solutions for the coupled BVP (3.1.1)-(3.1.2). Suppose that the functions ϕ and ψ are monotone nondecreasing and nonincreasing in the fifth and sixth arguments respectively. In addition, suppose that the functions*

$$\phi_\alpha(x,y) := \phi\left(\alpha_1(0),\alpha_2(0),x,y,\alpha_1^{'}(0),\alpha_2^{'}(0)\right),$$
$$\phi_\beta(x,y) := \phi\left(\beta_1(0),\beta_2(0),x,y,\beta_1^{'}(0),\beta_2^{'}(0)\right),$$

are monotone on $[\alpha_1(1), \beta_1(1)] \times [\alpha_2(1), \beta_2(1)]$ *and that the functions*

$$\psi_\alpha(x, y) := \psi\left(x, y, \alpha_1(1), \alpha_2(1), \alpha_1^{'}(1), \alpha_2^{'}(1)\right),$$

$$\psi_\beta(x, y) := \psi\left(x, y, \beta_1(1), \beta_2(1), \beta_1^{'}(1), \beta_2^{'}(1)\right),$$

are monotone on $[\alpha_1(0), \beta_1(0)] \times [\alpha_2(0), \beta_2(0)]$.

Then there exists at least one solution $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$ *of the coupled BVP* (3.1.1)-(3.1.2).

Proof. Let $\lambda > 0$ and consider the following modified coupled BVP

$$u_1^{''}(t) - \lambda u_1(t) = F_1^*(t, u_1(t), u_2(t)), \quad t \in [0, 1],$$
$$u_2^{''}(t) - \lambda u_2(t) = F_2^*(t, u_1(t), u_2(t)), \quad t \in [0, 1],$$
$$\phi^*\left(u_1(0), u_2(0), u_1(1), u_2(1), u_1^{'}(0), u_2^{'}(0)\right) = (u_1(0), u_2(0)),$$
$$\psi^*\left(u_1(0), u_2(0), u_1(1), u_2(1), u_1^{'}(1), u_2^{'}(1)\right) = (u_1(1), u_2(1)), \quad (3.1.26)$$

with

$$F_1^*(t, u_1(t), u_2(t)) = \begin{cases} f_1\left(t, \beta_2(t)\right) - \lambda\beta_1(t) & \text{if } (u_1(t), u_2(t)) \not\leq (\beta_1(t), \beta_2(t)), \\ f_1\left(t, u_2(t)\right) - \lambda\beta_1(t) & \text{if } \alpha_2(t) \leq u_2(t) \leq \beta_2(t), u_1(t) > \beta_1(t), \\ f_1\left(t, \alpha_2(t)\right) - \lambda\beta_1(t) & \text{if } u_2(t) < \alpha_2(t), u_1(t) > \beta_1(t), \\ f_1\left(t, \beta_2(t)\right) - \lambda u_1(t) & \text{if } u_2(t) > \beta_2(t), \alpha_1(t) \leq u_1(t) \leq \beta_1(t), \\ f_1\left(t, u_2(t)\right) - \lambda u_1(t) & \text{if } \alpha_2(t) \leq u_2(t) \leq \beta_2(t), \\ & \quad\quad \alpha_1(t) \leq u_1(t) \leq \beta_1(t), \\ f_1\left(t, \alpha_2(t)\right) - \lambda u_1(t) & \text{if } u_2(t) < \alpha_2(t), \alpha_1(t) \leq u_1(t) \leq \beta_1(t), \\ f_1\left(t, \beta_2(t)\right) - \lambda\alpha_1(t) & \text{if } u_2(t) > \beta_2(t), u_1(t) < \alpha_1(t), \\ f_1\left(t, u_2(t)\right) - \lambda\alpha_1(t) & \text{if } \alpha_2(t) \leq u_2(t) \leq \beta_2(t), u_1(t) < \alpha_1(t), \\ f_1\left(t, \alpha_2(t)\right) - \lambda\alpha_1(t) & \text{if } u_2(t) < \alpha_2(t), u_1(t) < \alpha_1(t), \end{cases}$$

and

$$F_2^*(t, u_1(t), u_2(t)) = \begin{cases} f_2\left(t, \beta_1(t)\right) - \lambda\beta_2(t) & \text{if } (u_1(t), u_2(t)) \not\leq (\beta_1(t), \beta_2(t)), \\ f_2\left(t, u_1(t)\right) - \lambda\beta_2(t) & \text{if } \alpha_1(t) \leq u_1(t) \leq \beta_1(t), u_2(t) > \beta_2(t), \\ f_2\left(t, \alpha_1(t)\right) - \lambda\beta_2(t) & \text{if } u_1(t) < \alpha_1(t), u_2(t) > \beta_2(t), \\ f_2\left(t, \beta_1(t)\right) - \lambda u_2(t) & \text{if } u_1(t) > \beta_1(t), \alpha_2(t) \leq u_2(t) \leq \beta_2(t), \\ f_2\left(t, u_1(t)\right) - \lambda u_2(t) & \text{if } \alpha_1(t) \leq u_1(t) \leq \beta_1(t), \\ & \quad\quad \alpha_2(t) \leq u_2(t) \leq \beta_2(t), \\ f_2\left(t, \alpha_1(t)\right) - \lambda u_2(t) & \text{if } u_1(t) < \alpha_1(t), \alpha_2(t) \leq u_2(t) \leq \beta_2(t), \\ f_2\left(t, \beta_1(t)\right) - \lambda\alpha_2(t) & \text{if } u_1(t) > \beta_1(t), u_2(t) < \alpha_2(t), \\ f_2\left(t, u_1(t)\right) - \lambda\alpha_2(t) & \text{if } \alpha_1(t) \leq u_1(t) \leq \beta_1(t), u_2(t) < \alpha_2(t), \\ f_2\left(t, \alpha_1(t)\right) - \lambda\alpha_2(t) & \text{if } u_1(t) < \alpha_1(t), u_2(t) < \alpha_2(t), \end{cases}$$

$$\phi^*(j, k, l, m, n, o) = p\left(0, (j, k) + \phi\left(j, k, l, m, n, o\right)\right),$$
$$\psi^*(j, k, l, m, n, o) = p\left(1, (l, m) + \psi\left(j, k, l, m, n, o\right)\right),$$

and

$$p(t,(x,y)) = \begin{cases} (\beta_1(t), \beta_2(t)) & \text{if } (x,y) \not\preceq (\beta_1(t), \beta_2(t)), \\ (x,y) & \text{if } (\alpha_1(t), \alpha_2(t)) \preceq (x,y) \preceq (\beta_1(t), \beta_2(t)), \\ (\alpha_1(t), \alpha_2(t)) & \text{if } (x,y) \not\succeq (\alpha_1(t), \alpha_2(t)), \end{cases}$$

Note that if $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$ is a solution of (3.1.26), then (u_1, u_2) is a solution of (3.1.1)-(3.1.2).

For the sake of simplicity we divide the proof in three steps:

Step 1: We define the mappings

$$L, N : C^2[0,1] \times C^2[0,1] \to C_0^2[0,1] \times C_0^2[0,1] \times \mathbb{R}^2 \times \mathbb{R}^2,$$

by

$$[L(u_1, u_2)](t) = \left(u_1'(t) - u_1'(0) - \lambda \int_0^t u_1(s)\,\mathrm{d}s, u_2'(t) - u_2'(0) - \lambda \int_0^t u_2(s)\,\mathrm{d}s, \right.$$
$$\left. (u_1(0), u_2(0)), (u_1(1), u_2(1)) \right),$$

and

$$[N(u_1, u_2)](t) = \left(\int_0^t F_1^*(s, u_1(s), u_2(s))ds, \int_0^t F_2^*(s, u_1(s), u_2(s))ds, \right.$$
$$\phi^*(u_1(0), u_2(0), u_1(1), u_2(1), u_1'(0), u_2'(0)),$$
$$\left. \psi^*(u_1(0), u_2(0), u_1(1), u_2(1), u_1'(1), u_2'(1)) \right).$$

Since $F_1^*(s, u_1(s), u_2(s))$ and $F_2^*(s, u_1(s), u_2(s))$ are bounded on $[0,1] \times \mathbb{R}^2$ and integral is a continuous function on $[0,1]$. Further ϕ^* and ψ^* being constant functions are continuous. Therefore $[N(u_1, u_2)]$ is continuous on $[0,1]$. Further, the class $\{N(u_1, u_2) : u_1, u_2 \in C^2[0,1]\}$ is uniformly bounded and equicontinuous. Therefore in view of Arzelà-Ascoli theorem $\{N(u_1, u_2) : u_1, u_2 \in C^2[0,1]\}$ is relatively compact. Consequently N is a compact map. Also from Lemma 3.1.1 with $a = 1, b = 0$, $c = 1, d = 0$ and $E = 0$, $F = 1$, $G = 0$, $H = 1$, L^{-1}, exists and is continuous.

On the other hand, solving (3.1.26) is equivalent to find a fixed point of

$$L^{-1}N : C^2[0,1] \times C^2[0,1] \to C^2[0,1] \times C^2[0,1].$$

Now, Schauder's fixed point theorem guarantees the existence of at least a fixed point since $L^{-1}N$ is continuous and compact.

Step 2: If (u_1, u_2) is a solution of (3.1.26), then $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$.

We claim $(u_1, u_2) \preceq (\beta_1, \beta_2)$. If $(u_1, u_2) \not\preceq (\beta_1, \beta_2)$, then either $u_1 \not\preceq \beta_1$ and/or $u_2 \not\preceq \beta_2$. If $u_1 \not\preceq \beta_1$, then $u_1 - \beta_1$ attains a positive maximum at some $t_0 \in [0,1]$. Clearly, $u_1(t_0) - \beta_1(t_0) > 0$. Thus $(u_1 - \beta_1)'(t_0) = 0$ and $(u_1 - \beta_1)''(t_0) < 0$. But,

$$(u_1 - \beta_1)''(t_0) > F_1^*(t_0, u_1(t_0), u_2(t_0)) + \lambda u_1(t_0) - f_1(t_0, \beta_2(t_0))$$
$$= f_1(t_0, \beta_2(t_0)) - \lambda \beta_1(t_0) + \lambda u_1(t_0) - f_1(t_0, \beta_2(t_0))$$
$$= \lambda(u_1(t_0) - \beta_1(t_0)) > 0,$$

a contradiction. Similarly, one can show that $(\alpha_1, \alpha_2) \preceq (u_1, u_2)$. Hence $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$.

Step 3: If (u_1, u_2) is a solution of (3.1.26), then (u_1, u_2) satisfies (3.1.2).

We claim

$$(\alpha_1(0), \alpha_2(0)) \preceq (u_1(0), u_2(0)) + \phi\left(u_1(0), u_2(0), u_1(1), u_2(1), u_1'(0), u_2'(0)\right) \preceq (\beta_1(0), \beta_2(0)).$$
(3.1.27)

If $(u_1(0), u_2(0)) + \phi\left(u_1(0), u_2(0), u_1(1), u_2(1), u_1'(0), u_2'(0)\right) \npreceq (\beta_1(0), \beta_2(0))$, then

$$
\begin{aligned}
(u_1(0), u_2(0)) &= \phi^*\left(u_1(0), u_2(0), u_1(1), u_2(1), u_1'(0), u_2'(0)\right) \\
&= p(0, (u_1(0), u_2(0)) + \phi(u_1(0), u_2(0), u_1(1), u_2(1), u_1'(0), u_2'(0))) \\
&= (\beta_1(0), \beta_2(0)).
\end{aligned}
$$

From Step 2, we know that $(u_1, u_2) \preceq (\beta_1, \beta_2)$, and this together with
$(u_1 - \beta_1, u_2 - \beta_2) \in C^2[0,1] \times C^2[0,1]$ and $(u_1(0), u_2(0)) = (\beta_1(0), \beta_2(0))$ yields $u_1'(0) \leq \beta_1'(0)$ and
$u_2'(0) \leq \beta_2'(0)$. If $\phi_\beta(x, y)$ is monotone nonincreasing, then we have

$$
\begin{aligned}
(u_1(0), u_2(0)) \;+\; & \phi\left(u_1(0), u_2(0), u_1(1), u_2(1), u_1'(0), u_2'(0)\right) \\
=\; & (\beta_1(0), \beta_2(0)) + \phi\left(\beta_1(0), \beta_2(0), u_1(1), u_2(1), u_1'(0), u_2'(0)\right) \\
\preceq\; & (\beta_1(0), \beta_2(0)) + \phi\left(\beta_1(0), \beta_2(0), u_1(1), u_2(1), \beta_1'(0), \beta_2'(0)\right) \\
=\; & (\beta_1(0), \beta_2(0)) + \phi_\beta\left(u_1(1), u_2(1)\right) \\
\preceq\; & (\beta_1(0), \beta_2(0)) + \phi_\beta\left(\alpha_1(1), \alpha_2(1)\right) \\
=\; & (\beta_1(0), \beta_2(0)) + \phi\left(\beta_1(0), \beta_2(0), \alpha_1(1), \alpha_2(1), \beta_1'(0), \beta_2'(0)\right) \\
\preceq\; & (\beta_1(0), \beta_2(0)),
\end{aligned}
$$
(3.1.28)

a contradiction. Similarly, if $\phi_\beta(x, y)$ is monotone nondecreasing, then we get same contradiction. Consequently, (3.1.27) holds. Similar reasoning shows the other boundary condition. Consequently, (u_1, u_2) satisfies (3.1.2). Hence the system of the coupled BVPs (3.1.1)-(3.1.2) has at-least one solution $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$.

3.1.4 Application of the theoretical results

In this section we study some examples to demonstrate the validity of our theoretical results.

Example 3.1.1. *Consider the following nonlinear second order coupled system*

$$
\begin{aligned}
u_1''(t) &= u_2(t) + \sin(t), \quad t \in [0, 1], \\
u_2''(t) &= u_1^3(t) + \cos(t), \quad t \in [0, 1],
\end{aligned}
$$
(3.1.29)

with the following nonlinear CBCs

$$
\left(u_1^{\frac{1}{2n+1}}(1) - u_2(0), u_2^{\frac{1}{2n+1}}(1) - u_1(0)\right) = (0, 0), \quad n \in \mathbb{N},
$$

$$
\left(u_1'(1) - u_2^{2n+1}(1), u_2'(1) - u_1^{2n+1}(1)\right) = (0, 0), \quad n \in \mathbb{N}.
$$
(3.1.30)

Choose $\alpha_1(t) = \sin(t) - 5$, $\alpha_2(t) = \sin(t) - 6$ and $\beta_1(t) = \sin(t) + 5$, $\beta_2(t) = \sin(t) + 6$. In view of the Definition 3.1.1, it is evident that a couple of functions (α_1, α_2) and (β_1, β_2) are lower and

upper solutions of the coupled system (3.1.29), *respectively. Further,* (α_1, α_2) *and* (β_1, β_2) *satisfies the system of inequalities* (3.1.25). *Hence by Theorem 3.1.2, the coupled system of BVPs* (3.1.29)-(3.1.30) *has at least one solution* $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$.

Example 3.1.2. *Consider the following nonlinear second order coupled system*

$$u_1^{''}(t) = u_2^3(t) - 10\sqrt{\sin(t)}, \qquad t \in [0, 1],$$
$$u_2^{''}(t) = u_1^3(t) - 8\sqrt{\cos(t-1)}, \quad t \in [0, 1], \tag{3.1.31}$$

with the following nonlinear CBCs

$$\left(u_1(0)u_1(1) - u_2(0)u_2(1), u_1(0)u_1^{'}(0) - u_2(0)u_2^{'}(0)\right) = (0, 0),$$
$$\left(u_1(0)u_1^{'}(1) - u_2(0)u_2(1), u_1(1)u_1^{'}(1) - u_2(1)u_2^{'}(1)\right) = (0, 0). \tag{3.1.32}$$

Choose $\alpha_1(t) = -\frac{t^3}{2}$, $\alpha_2(t) = -\frac{t^5}{3}$ *and* $\beta_1(t) = t + 2$, $\beta_2(t) = t + 3$. *In view of the Definition 3.1.1, it is evident that a couple of functions* (α_1, α_2) *and* (β_1, β_2) *are lower and upper solutions of the coupled system* (3.1.31), *respectively. Further,* (α_1, α_2) *and* (β_1, β_2) *satisfies the system of inequalities* (3.1.25). *Hence by Theorem 3.1.2, the coupled system of BVPs* (3.1.31)-(3.1.32) *has at least one solution* $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$.

$\square$

3.2 Second-order coupled system with nonlinear coupled boundary conditions (II)

In this section, we study the existence of solutions of the following second-order ordinary nonlinear coupled system of BVPs of the type

$$-u_1^{''}(t) = f_1(t, u_2(t)), \quad t \in [0, 1],$$
$$-u_2^{''}(t) = f_2(t, u_1(t)), \quad t \in [0, 1], \tag{3.2.1}$$

subject to nonlinear CBCs

$$\mu\left(u_1(0), u_2(0), u_1^{'}(0), u_2^{'}(0), u_1^{'}(1), u_2^{'}(1)\right) = (0, 0),$$
$$\nu\left(u_1(0), u_2(0)\right) + (u_1(1), u_2(1)) = (0, 0), \tag{3.2.2}$$

where $f_1, f_2 : [0, 1] \times \mathbb{R} \to \mathbb{R}$, $\mu : \mathbb{R}^6 \to \mathbb{R}^2$ and $\nu : \mathbb{R}^2 \to \mathbb{R}^2$ are continuous functions. We establish the existence of at-least one solution of (3.2.1)–(3.2.2) with the help of LUSs method in the presence of the existence of well ordered LUSs such that $(\alpha_1, \alpha_2) \preceq (\beta_1, \beta_2)$ on $[0, 1] \times [0, 1]$.

The monotonicity assumptions on ϕ and ψ as stated in Theorem 3.1.2 does not address to the existence criterion for a very important BVPs like periodic and anti-periodic. But in this study,

we introduce a new generalized BCs 3.2.2 that satisfy the existence criterion for a periodic and anti-periodic BVPs as a particular case under some monotonicity assumptions on the arguments of the boundary functions μ and ν as discussed in detail in the introductory chapter of this thesis.

3.2.1 Lower and upper solutions method

In this section we recall the idea of LUSs for the nonlinear coupled system (3.2.1).

Definition 3.2.1. We say that a couple of function $(\alpha_1, \alpha_2) \in C^2[0,1] \times C^2[0,1]$ is a lower solution of (3.2.1) if

$$
\begin{aligned}
-\alpha_1''(t) &\leq f_1\left(t, \alpha_2(t)\right), \quad t \in [0,1], \\
-\alpha_2''(t) &\leq f_2\left(t, \alpha_1(t)\right), \quad t \in [0,1].
\end{aligned}
\tag{3.2.3}
$$

In the same way, an upper solution is a function $(\beta_1, \beta_2) \in C^2[0,1] \times C^2[0,1]$, that satisfies the same inequalities in reverse order. For $u_1, u_2 \in C^2[0,1]$, we define the set

$$
[\![u_1, u_2]\!] = \left\{ w_2 \in C^2[0,1] : u_1(t) \leq w_2(t) \leq u_2(t), t \in [0,1] \right\}.
$$

3.2.2 Coupled lower and upper solutions

This section deals to the concept of CLUSs that unifies the treatment of many linear and nonlinear BVPs under some monotonicity assumptions on the arguments of the boundary functions μ and ν.

Definition 3.2.2. We say that a couple of functions (α_1, α_2), $(\beta_1, \beta_2) \in C^2[0,1] \times C^2[0,1]$ are coupled lower and upper solutions for the problem (3.2.1) and (3.2.2), if (α_1, α_2) is a lower solution and (β_1, β_2) is an upper solution for the system (3.2.1),$(\alpha_1, \alpha_2) \preceq (\beta_1, \beta_2)$, if $\alpha_1 \leq \beta_1$ and $\alpha_2 \leq \beta_2$, and

$$
\begin{aligned}
\mu(\beta_1(0), \beta_2(0), \beta_1'(0), \beta_2'(0), \beta_1'(1), \beta_2'(1)) &\preceq (0,0) \preceq \mu(\alpha_1(0), \alpha_2(0), \alpha_1'(0), \alpha_2'(0), \alpha_1'(1), \alpha_2'(1)), \\
\mu(\beta_1(0), \beta_2(0), \beta_1'(0), \beta_2'(0), \alpha_1'(1), \alpha_2'(1)) &\preceq (0,0) \preceq \mu(\alpha_1(0), \alpha_2(0), \alpha_1'(0), \alpha_2'(0), \beta_1'(1), \beta_2'(1)), \\
(\alpha_1(1), \alpha_2(1)) + \nu\left(\beta_1(0), \beta_2(0)\right) &= (0,0), \\
(\beta_1(1), \beta_2(1)) + \nu\left(\alpha_1(0), \alpha_2(0)\right) &= (0,0), \\
(\alpha_1(1), \alpha_2(1)) + \nu\left(\alpha_1(0), \alpha_2(0)\right) &= (0,0), \\
(\beta_1(1), \beta_2(1)) + \nu\left(\beta_1(0), \beta_2(0)\right) &= (0,0).
\end{aligned}
\tag{3.2.4}
$$

3.2.3 Existence of at least one solution

Theorem 3.2.1. *Assume that a couple of functions (α_1, α_2) and (β_1, β_2) are coupled lower and upper solutions for the coupled BVPs (3.2.1)-(3.2.2). Suppose that μ is nondecreasing in the third and fourth arguments. In addition suppose that the function ν in $[\alpha_1(0), \beta_1(0)] \times [\alpha_2(0), \beta_2(0)]$ is monotone and the functions*

$$
\begin{aligned}
\mu_{(\alpha_1, \alpha_2)}(x, y) &:= \mu\left(\alpha_1(0), \alpha_2(0), \alpha_1'(0), \alpha_2'(0), x, y\right), \\
\mu_{(\beta_1, \beta_2)}(x, y) &:= \mu\left(\beta_1(0), \beta_2(0), \beta_1'(0), \beta_2'(0), x, y\right),
\end{aligned}
$$

have got the same kind of monotonicity as ν, then there exists at least one solution $(u_1, u_2) \in$ $[\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$ of the coupled BVP (3.2.1)-(3.2.2).

Proof. Let $\lambda > 0$, and consider the following modified coupled BVP

$$-u_1''(t) + \lambda u_1(t) = F_1^* (t, u_1(t), u_2(t)), \quad t \in [0, 1],$$
$$-u_2''(t) + \lambda u_2(t) = F_2^* (t, u_1(t), u_2(t)), \quad t \in [0, 1],$$
$$\mu^* \left(u_1(0), u_2(0), u_1(1), u_2(1), u_1'(0), u_2'(0) \right) = (u_1(0), u_2(0)),$$
$$(u_1(1), u_2(1)) + \nu^* (u_1(0), u_2(0)) = (0, 0), \tag{3.2.5}$$

with

$$F_1^*(t, u_1(t), u_2(t)) = \begin{cases} f_1(t, \beta_2(t)) + \lambda\beta_1(t) & \text{if } (u_1(t), u_2(t)) \not\preceq (\beta_1(t), \beta_2(t)), \\ f_1(t, u_2(t)) + \lambda\beta_1(t) & \text{if } \alpha_2(t) \le u_2(t) \le \beta_2(t), u_1(t) > \beta_1(t), \\ f_1(t, \alpha_2(t)) + \lambda\beta_1(t) & \text{if } u_2(t) < \alpha_2(t), u_1(t) > \beta_1(t), \\ f_1(t, \beta_2(t)) + \lambda u_1(t) & \text{if } u_2(t) > \beta_2(t), \alpha_1(t) \le u_1(t) \le \beta_1(t), \\ f_1(t, u_2(t)) + \lambda u_1(t) & \text{if } \alpha_2(t) \le u_2(t) \le \beta_2(t), \\ & \quad \alpha_1(t) \le u_1(t) \le \beta_1(t), \\ f_1(t, \alpha_2(t)) + \lambda u_1(t) & \text{if } u_2(t) < \alpha_2(t), \alpha_1(t) \le u_1(t) \le \beta_1(t), \\ f_1(t, \beta_2(t)) + \lambda\alpha_1(t) & \text{if } u_2(t) > \beta_2(t), u_1(t) < \alpha_1(t), \\ f_1(t, u_2(t)) + \lambda\alpha_1(t) & \text{if } \alpha_2(t) \le u_2(t) \le \beta_2(t), u_1(t) < \alpha_1(t), \\ f_1(t, \alpha_2(t)) + \lambda\alpha_1(t) & \text{if } u_2(t) < \alpha_2(t), u_1(t) < \alpha_1(t), \end{cases}$$

and

$$F_2^*(t, u_1(t), u_2(t)) = \begin{cases} f_2(t, \beta_1(t)) + \lambda\beta_2(t) & \text{if } (u_1(t), u_2(t)) \not\preceq (\beta_1(t), \beta_2(t)), \\ f_2(t, u_1(t)) + \lambda\beta_2(t) & \text{if } \alpha_1(t) \le u_1(t) \le \beta_1(t), u_2(t) > \beta_2(t), \\ f_2(t, \alpha_1(t)) + \lambda\beta_2(t) & \text{if } u_1(t) < \alpha_1(t), u_2(t) > \beta_2(t), \\ f_2(t, \beta_1(t)) + \lambda u_2(t) & \text{if } u_1(t) > \beta_1(t), \alpha_2(t) \le u_2(t) \le \beta_2(t), \\ f_2(t, u_1(t)) + \lambda u_2(t) & \text{if } \alpha_1(t) \le u_1(t) \le \beta_1(t), \\ & \quad \alpha_2(t) \le u_2(t) \le \beta_2(t), \\ f_2(t, \alpha_1(t)) + \lambda u_2(t) & \text{if } u_1(t) < \alpha_1(t), \alpha_2(t) \le u_2(t) \le \beta_2(t), \\ f_2(t, \beta_1(t)) + \lambda\alpha_2(t) & \text{if } u_1(t) > \beta_1(t), u_2(t) < \alpha_2(t), \\ f_2(t, u_1(t)) + \lambda\alpha_2(t) & \text{if } \alpha_1(t) \le u_1(t) \le \beta_1(t), u_2(t) < \alpha_2(t), \\ f_2(t, \alpha_1(t)) + \lambda\alpha_2(t) & \text{if } u_1(t) < \alpha_1(t), u_2(t) < \alpha_2(t), \end{cases}$$

$$\mu^* (j, k, l, m, n, o) = \phi (0, (j, k) + \mu (j, k, l, m, n, o)),$$
$$\nu^* (j, k) = \nu (\phi (0, (j, k))),$$

$$\phi(t, (x, y)) = \begin{cases} (\beta_1(t), \beta_2(t)) & \text{if } (x, y) \not\preceq (\beta_1, \beta_2) \\ (x, y) & \text{if } (\alpha_1, \alpha_2) \preceq (x, y) \preceq (\beta_1, \beta_2) \\ (\alpha_1(t), \alpha_2(t)) & \text{if } (x, y) \not\succeq (\alpha_1, \alpha_2), \end{cases}$$

Note that if $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$ is a solution of (3.2.5), then (u_1, u_2) is a solution of (3.2.1)-(3.2.2).

For the sake of simplicity we divide the proof in three steps:

Step 1: We define the mappings

$$L, N : C^2[0,1] \times C^2[0,1] \to C_0^2[0,1] \times C_0^2[0,1] \times \mathbb{R}^2 \times \mathbb{R}^2,$$

by

$$
\begin{aligned}
[L(u_1, u_2)](t) \;=\; & \left(u_1'(t) - u_1'(0) - \lambda \int_0^t u_1(s)\, \mathrm{d}s,\, u_2'(t) - u_2'(0) - \lambda \int_0^t u_2(s)\, \mathrm{d}s, \right. \\
& \left. (u_1(0), u_2(0)),\, (u_1(1), u_2(1)) \right),
\end{aligned}
$$

and

$$
\begin{aligned}
[N(u_1, u_2)](t) \;=\; & \left(\int_0^t F_1^*(s, u_1(s), u_2(s))ds,\, \int_0^t F_2^*(s, u_1(s), u_2(s))ds, \right. \\
& \mu^*(u_1(0), u_2(0), u_1(1), u_2(1), u_1'(0), u_2'(0)), \\
& \left. -\nu^*(u_1(0), u_2(0)) \right).
\end{aligned}
$$

Since $F_1^*(s, u_1(s), u_2(s))$ and $F_2^*(s, u_1(s), u_2(s))$ are bounded on $[0,1] \times \mathbb{R}^2$ and integral is a continuous function on $[0,1]$. Further μ^* and ν^* being constant functions are continuous. Therefore $[N(u_1, u_2)]$ is continuous on $[0,1]$. Further, the class $\{N(u_1, u_2) : u_1, u_2 \in C^2[0,1]\}$ is uniformly bounded and equicontinuous. Therefore in view of Arzelà-Ascoli theorem $\{N(u_1, u_2) : u_1, u_2 \in C^2[0,1]\}$ is relatively compact. Consequently N is a compact map. Also from Lemma 3.1.1 with $a = 1, b = 0$, $c = 1, d = 0$ and $E = 0, F = 1, G = 0, H = 1$, L^{-1}, exists and is continuous.

On the other hand, solving (3.2.5) is equivalent to find a fixed point of

$$L^{-1}N : C^2[0,1] \times C^2[0,1] \to C^2[0,1] \times C^2[0,1].$$

Now, Schauder's fixed point theorem guarantees the existence of at least a fixed point since $L^{-1}N$ is continuous and compact.

Step 2: If (u_1, u_2) is a solution of (3.2.5), then $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$. By definition of μ^*, we see that $(u_1(0), u_2(0)) \in [\alpha_1(0), \beta_1(0)] \times [\alpha_2(0), \beta_2(0)]$. Thus, if ν is nondecreasing, we have by condition (3.2.4)

$$
\begin{aligned}
(\alpha_1(1), \alpha_2(1)) = -\nu(\beta_1(0), \beta_2(0)) &\preceq -\nu(u_1(0), u_2(0)) = (u_1(1), u_2(1)) \preceq -\nu(\alpha_1(0), \alpha_2(0)) \\
(\alpha_1(1), \alpha_2(1)) &\preceq (u_1(1), u_2(1)) \preceq (\beta_1(1), \beta_2(1)),
\end{aligned} \tag{3.2.6}
$$

similarly, if ν is nonincreasing, then (3.2.6) holds. Hence $(u_1(1), u_2(1)) \in [\alpha_1(1), \beta_1(1)] \times [\alpha_2(1), \beta_2(1)]$. Now, it remains to show that $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$ for $t \in (0,1)$.

We claim $(u_1, u_2) \preceq (\beta_1, \beta_2)$. If $(u_1, u_2) \npreceq (\beta_1, \beta_2)$, then $u_1 \npreceq \beta_1$ and/or $u_2 \npreceq \beta_2$. If $u_1 \npreceq \beta_1$, then there exists some $t_0 \in [0,1]$ such that $u_1(t_0) - \beta_1(t_0) > 0$. So, $u_1 - \beta_1$ attains a positive maximum at $t_0 \in [0,1]$. Thus $(u_1 - \beta_1)'(t_0) = 0$ and $(u_1 - \beta_1)''(t_0) < 0$. But,

$$
\begin{aligned}
(u_1 - \beta_1)''(t_0) &> -F_1^*(t_0, u_1(t_0), u_2(t_0)) + \lambda u_1(t_0) + f_1(t_0, \beta_2(t_0)) \\
&= -f_1(t_0, \beta_2(t_0)) - \lambda \beta_1(t_0) + \lambda u_1(t_0) + f_1(t_0, \beta_2(t_0)) \\
&= \lambda(u_1(t_0) - \beta_1(t_0)) > 0,
\end{aligned}
$$

a contradiction. Similarly one can show that $(\alpha_1, \alpha_2) \preceq (u_1, u_2)$. Hence $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$.

Step 3: If (u_1, u_2) is a solution of (3.2.5) then (u_1, u_2) satisfies (3.2.2).

We claim

$$(\alpha_1(0), \alpha_2(0)) \preceq (u_1(0), u_2(0)) + \mu\left(u_1(0), u_2(0), u_1'(0), u_2'(0), u_1'(1), u_2'(1)\right) \preceq (\beta_1(0), \beta_2(0)). \quad (3.2.7)$$

If $(\alpha_1(0), \alpha_2(0)) \npreceq (u_1(0), u_2(0)) + \mu\left(u_1(0), u_2(0), u_1'(0), u_2'(0), u_1'(1), u_2'(1)\right)$, then

$$\begin{aligned}
(u_1(0), u_2(0)) &= \mu^*\left(u_1(0), u_2(0), u_1'(0), u_2'(0), u_1'(1), u_2'(1)\right) \\
&= \phi(0, (u_1(0), u_2(0)) + \mu(u_1(0), u_2(0), u_1'(0), u_2'(0), u_1'(1), u_2'(1))) \\
&= (\alpha_1(0), \alpha_2(0)).
\end{aligned} \quad (3.2.8)$$

If ν is nondecreasing, we have

$$(u_1(1), u_2(1)) = -\nu\left(u_1(0), u_2(0)\right) = -\nu\left(\alpha_1(0), \alpha_2(0)\right) = (\beta_1(1), \beta_2(1)). \quad (3.2.9)$$

Using (3.2.8),(3.2.9) and Step 2, we have $\left(u_1'(0), u_2'(0)\right) \succeq \left(\alpha_1'(0), \alpha_2'(0)\right)$ and $\left(u_1'(1), u_2'(1)\right) \succeq \left(\beta_1'(1), \beta_2'(1)\right)$. But

$$\begin{aligned}
(u_1(0), u_2(0)) \quad &+ \quad \mu\left(u_1(0), u_2(0), u_1'(0), u_2'(0), u_1'(1), u_2'(1)\right) \\
&= \quad (\alpha_1(0), \alpha_2(0)) + \mu\left(\alpha_1(0), \alpha_2(0), u_1'(0), u_2'(0), u_1'(1), u_2'(1)\right) \\
&\succeq \quad (\alpha_1(0), \alpha_2(0)) + \mu\left(\alpha_1(0), \alpha_2(0), \alpha_1'(0), \alpha_2'(0), u_1'(1), u_2'(1)\right) \\
&= \quad (\alpha_1(0), \alpha_2(0)) + \mu_{(\alpha_1, \alpha_2)}\left(u_1'(1), u_2'(1)\right) \\
&\succeq \quad (\alpha_1(0), \alpha_2(0)) + \mu\left(\alpha_1(0), \alpha_2(0), \alpha_1'(0), \alpha_2'(0), \beta_1'(1), \beta_2'(1)\right) \\
&\succeq \quad (\alpha_1(0), \alpha_2(0)),
\end{aligned} \quad (3.2.10)$$

a contradiction. Similarly if ν is nonincreasing we get same contradiction. Consequently, (3.2.7) holds. By definition of ν^* and Step 2, the second boundary condition is obvious. Consequently (u_1, u_2) satisfies (3.2.2). Hence the coupled system of BVPs (3.2.1)-(3.2.2) has at least one solution $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$. $\qquad\square$

3.2.4 Application of the theoretical results

Example 3.2.1. *Consider the following nonlinear coupled system of BVP*

$$\begin{aligned}
-u_1''(t) &= u_2^5(t) - 20\sqrt{\sin(t)}, & t &\in [0, 1], \\
-u_2''(t) &= u_1^5(t) - 16\sqrt{\cos(t-1)}, & t &\in [0, 1],
\end{aligned} \quad (3.2.11)$$

with the following nonlinear CBCs

$$\begin{aligned}
&(u_1(0)u_1'(0) - u_2(0)u_2'(0),\ u_1(0)u_1'(1) - u_2'(0)u_2'(1)) = (0, 0), \\
&\left(\sqrt{\tan(u_1(0))}u_1(1) + \tan(u_2(0))u_2(1),\ \sqrt{\tan(u_1(0))} + \tan(u_2(0))u_2(1)\right) = (0, 0).
\end{aligned} \quad (3.2.12)$$

Choose $\alpha_1(t) = -\frac{t^5}{4}$, $\alpha_2(t) = -\frac{t^7}{5}$ *and* $\beta_1(t) = \frac{t^5}{4}$, $\beta_2(t) = \frac{t^7}{5}$. *In the light of the Definition 3.2.1, It is evident that a couple of functions* (α_1, α_2) *and* (β_1, β_2) *are lower and upper solutions of the coupled system (3.2.11), respectively. Further,* (α_1, α_2) *and* (β_1, β_2) *satisfy (3.2.4). Hence by Theorem 3.2.1, the coupled system of the BVPs (3.2.11)-(3.2.12) has at least one solution* $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$.

Example 3.2.2. *Consider the following nonlinear coupled system of BVP*

$$
\begin{aligned}
&- u_1^{''}(t) = t^2 + \sin(u_2(t)), \quad t \in [0, 1], \\
&- u_2^{''}(t) = 5t^2 + \cos(u_1(t) + 4) + \sin(u_1(t) - 4), \quad t \in [0, 1],
\end{aligned}
\tag{3.2.13}
$$

with the following nonlinear CBCs

$$
\begin{aligned}
&(u_1(0)u_1'(1) - u_2'(0)u_2'(1), u_1'(0)u_1'(1) - u_2'(1)) = (0, 0), \\
&(\sin(u_1(0)) + u_1(0)u_2(1), \sin(u_2(0)) + u_2(0)u_1(1)) = (0, 0).
\end{aligned}
\tag{3.2.14}
$$

Choose $\alpha_1(t) = t^2$, $\alpha_2(t) = t^3$ *and* $\beta_1(t) = t$, $\beta_2(t) = t$. *In the light of the Definition 3.2.1, It is evident that a couple of functions* (α_1, α_2) *and* (β_1, β_2) *are lower and upper solutions of the system (3.2.13), respectively. Further,* (α_1, α_2) *and* (β_1, β_2) *satisfy the coupled system (3.2.4). Hence by Theorem 3.2.1, the system of BVPs (3.2.13)-(3.2.14) has at least one solution* $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$.

Chapter 4

Nonlinear Coupled Boundary Value Problems With Derivative Dependent Nonlinearity

In this chapter, we extend the results of Chapter 3 for a nonlinear second-order coupled BVPs subject to nonlinear CBCs with dependence of the nonlinearity on derivative. The existence results are established assuming the existence of well ordered LUSs, such that $(\alpha_1, \alpha_2) \preceq (\beta_1, \beta_2)$, if $\alpha_1 \leq \beta_1$ and $\alpha_2 \leq \beta_2$ on $[0, 1]$ under Nagumo conditions. The classical LUSs approach is adopted to ensure the existence of at least one solution in the closed interval defined by a well ordered LUSs.

The study of BVPs with nonlinearity dependent on the derivative with certain type of BCs was initially studied by Schmitt in [167] and developed the classical results to ensure the existence of solutions of these types of BVPs using LUSs approach under Nagumo condition. Currently, extensive study has been done on these types of BVPs including, Dirichlet, periodic, and multi-point BCs, see for example [55, 80, 91, 110, 111, 166, 175, 189, 194] and Chapter III of [102]. In [13], Chapter III, IV, the new classical results are established with multi-point linear coupled BCs for the case when the derivative dependent nonlinear terms are allowed to be a singular at the end points of the closed interval $[0, 1]$. In contrast of these studies we develop the existence results for derivative dependent nonlinear coupled BVPs corresponding to a generalized nonlinear CBCs in the presence of the Nagumo condition.

This chapter is organized as: In Section 4.1.1, the idea of LUSs is recalled for the second-order nonlinear coupled BVPs having dependence of the nonlinearity on the first-order derivative. In

Section 4.1.2, the Nagumo conditions are presented that ensure the existence of the solutions of the BVPs having dependence of the nonlinearity on the derivative. In Section 4.1.3, the idea of CLUSs is discussed. In Section 4.1.4, the existence of at-least one solution is ensured using LUSs method. In Section 4.1.5, an example is considered to check the validity of the developed theoretical results. In Section 4.2.1, the concept of CLUSs is discussed for the more generalized BCs. In Section 4.2.2, the LUSs methodology is used to investigate the existence of at-least one solution for the second-order BVPs with dependence of the nonlinear terms on the first-order derivative. In Section 4.2.3, an illustrative example is discussed to check the validity of the developed theoretical result.

4.1 Second-order coupled system with derivative dependent nonlinearity (I)

In this section, we discuss the existence of solutions of the following second-order nonlinear coupled system of BVPs having nonlinearity in the first-order derivative of the type

$$-u_1^{''}(t) = f_1(t, u_1(t), u_2(t), u_1^{'}(t), u_2^{'}(t)), \quad t \in [0,1],$$
$$-u_2^{''}(t) = f_2(t, u_1(t), u_2(t), u_1^{'}(t), u_2^{'}(t)), \quad t \in [0,1], \tag{4.1.1}$$

$$\phi(u_1(0), u_2(0), u_1(1), u_2(1), u_1^{'}(0), u_2^{'}(0)) = (0,0),$$
$$\psi(u_1(0), u_2(0), u_1(1), u_2(1), u_1^{'}(1), u_2^{'}(1)) = (0,0), \tag{4.1.2}$$

where $f_1, f_2 : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $\phi, \psi : \mathbb{R}^6 \to \mathbb{R}^2$ are continuous functions.

The BCs (4.1.2) are very much generalized in the sense that they generalize most of the linear and nonlinear BCs under some monotonicity assumptions on the arguments of the boundary functions ϕ and ψ as discussed in detail in the introductory chapter of this thesis. Moreover (4.1.2) handle the classical existence criterion for the Dirichlet and Neumann BCs as a particular case.

4.1.1 Lower and upper solutions method

In this section, we recall the concepts of LUSs for the nonlinear coupled BVPs of the type (4.1.1) and Nagumo condition.

Definition 4.1.1. We say that a couple of function $(\alpha_1, \alpha_2) \in C^2[0,1] \times C^2[0,1]$ is a lower solution of (4.1.1) if

$$-\alpha_1^{''}(t) \le f_1\left(t, \alpha_1(t), \alpha_2(t), \alpha_1^{'}(t), \alpha_2^{'}(t)\right), \quad t \in [0,1],$$
$$-\alpha_2^{''}(t) \le f_2\left(t, \alpha_1(t), \alpha_2(t), \alpha_1^{'}(t), \alpha_2^{'}(t)\right), \quad t \in [0,1]. \tag{4.1.3}$$

In the same way, an upper solution is a function $(\beta_1, \beta_2) \in C^2[0,1] \times C^2[0,1]$, that satisfies the same inequalities in reverse order. For $u_1, u_2 \in C^2[0,1]$, we define the set

$$[\![u_1, u_2]\!] = \left\{w_3 \in C^2[0,1] : u_1(t) \le w_3(t) \le u_2(t), t \in [0,1]\right\}.$$

4.1.2   Nagumo condition

In 1954, Nagumo [137] observed that existence of LUSs is not sufficient to ensure the existence of solutions of Dirichlet BVPs when nonlinear terms involving the first-order derivative of the solution. This is due to the fact that nonlinear term grows very rapidly with respect to the gradient of the solution. Later on Habets and Pouse [87] observed the same problem for periodic and separated BVPs and found that method of LUSs is not valid without considering the Nagumo condition. Therefore to ensure the existence of solutions of the system (4.1.1), we need to find the prior bounds on u_1' and u_2'.

Now we define the concept of Nagumo condition for the nonlinear second order coupled system (4.1.1).

Definition 4.1.2. We say that f_1 and f_2 satisfies a Nagumo condition relative to the intervals $[\alpha_1(t), \beta_1(t)]$ and $[\alpha_2(t), \beta_2(t)]$ respectively, if for

$$p_0 = \max\left\{\beta_1(0) - \alpha_1(1) + \beta_2(0) - \alpha_2(1), \beta_1(1) - \alpha_1(0) + \beta_2(1) - \alpha_2(0)\right\},$$

there exists a constant N_i such that

$$N_i > \max\left\{p_0, \sup_{t \in [0,1]} |\alpha_i^{'}(t)|, \sup_{t \in [0,1]} |\beta_i^{'}(t)|\right\}, \quad i = 1, 2,$$

and a continuous function $g_1 : [0, \infty) \to (0, \infty)$ such that

$$|f_1(t, u_1(t), u_2(t), d, e| \le g_1(|d + e|), \quad t \in [0,1],$$
$$|f_2(t, u_1(t), u_2(t), d, e| \le g_2(|d + e|), \quad t \in [0,1], \tag{4.1.4}$$

whenever $(\alpha_1(t), \alpha_2(t)) \preceq (u_1(t), u_2(t)) \preceq (\beta_1(t), \beta_2(t)), \ d, e \in \mathbb{R}$, and

$$\int_{p_0}^{N_i} \frac{1}{g_1(y)} dy > 2.$$

4.1.3 Coupled lower and upper solutions

In this section we define the concept of CLUSs that unify the treatment of many linear and nonlinear BVPs by imposing some monotonicity assumptions on the arguments of the boundary functions ϕ and ψ.

Definition 4.1.3. We say that a couple of functions $((\alpha_1, \alpha_2) \text{ and } (\beta_1, \beta_2)) \in C^2[0,1] \times C^2[0,1]$ are coupled lower and upper solutions for the coupled BVP (4.1.1)–(4.1.2), if (α_1, α_2) is a lower solution and (β_1, β_2) is an upper solution for the system (4.1.1) with $(\alpha_1, \alpha_2) \preceq (\beta_1, \beta_2)$, if $\alpha_1 \leq \beta_1$ and $\alpha_2 \leq \beta_2$, such that

$$\phi(\beta_1(0), \beta_2(0), \beta_1(1), \beta_2(1), \beta_1'(0), \beta_2'(0)) \preceq (0,0) \preceq \phi(\alpha_1(0), \alpha_2(0), \alpha_1(1), \alpha_2(1), \alpha_1'(0), \alpha_2'(0)),$$
$$\phi(\beta_1(0), \beta_2(0), \alpha_1(1), \alpha_2(1), \beta_1'(0), \beta_2'(0)) \preceq (0,0) \preceq \phi(\alpha_1(0), \alpha_2(0), \beta_1(1), \beta_2(1), \alpha_1'(0), \alpha_2'(0)),$$
$$\psi(\beta_1(0), \beta_2(0), \beta_1(1), \beta_2(1), \beta_1'(1), \beta_2'(1)) \preceq (0,0) \preceq \psi(\alpha_1(0), \alpha_2(0), \alpha_1(1), \alpha_2(1), \alpha_1'(1), \alpha_2'(1)),$$
$$\psi(\alpha_1(0), \alpha_2(0), \beta_1(1), \beta_2(1), \beta_1'(1), \beta_2'(1)) \preceq (0,0) \preceq \psi(\beta_1(0), \beta_2(0), \alpha_1(1), \alpha_2(1), \alpha_1'(1), \alpha_2'(1)).$$

$$(4.1.5)$$

Now we state and prove theorem that establishes the existence of at least one solution for BVPs (4.1.1)-(4.1.2).

4.1.4 Existence of at least one solution

Theorem 4.1.1. *Suppose the nonlinear coupled BVP (4.1.1)-(4.1.2) having (α_1, α_2) and (β_1, β_2) coupled lower and upper solutions respectively and the nonlinear functions f_1 and f_2 relative to the intervals $[\alpha_1(t), \beta_1(t)]$ and $[\alpha_2(t), \beta_2(t)]$ satisfy the Nagumo assumptions respectively. Moreover, assume that ψ and ϕ are non-increasing and non-decreasing in the fifth and sixth arguments respectively and*

$$\psi_\alpha(x,y) := \psi\left(x, y, \alpha_1(1), \alpha_2(1), \alpha_1'(1), \alpha_2'(1)\right),$$
$$\psi_\beta(x,y) := \psi\left(x, y, \beta_1(1), \beta_2(1), \beta_1'(1), \beta_2'(1)\right),$$

are monotone on $[\alpha_1(0), \beta_1(0)] \times [\alpha_2(0), \beta_2(0)]$ and

$$\psi_\alpha(x,y) := \psi\left(\alpha_1(0), \alpha_2(0), x, y, \alpha_1'(0), \alpha_2'(0)\right),$$
$$\psi_\beta(x,y) := \psi\left(\beta_1(0), \beta_2(0), x, y, \beta_1'(0), \beta_2'(0)\right),$$

are monotone on $[\alpha_1(1), \beta_1(1)] \times [\alpha_2(1), \beta_2(1)]$.

Then the nonlinear coupled system of BVPs (4.1.1)-(4.1.2) have at least one solution $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$ and $(-N_1, -N_2) \preceq (u_1'(t), u_2'(t)) \preceq (N_1, N_2), t \in [0,1]$.

Proof. Let $\lambda > 0$, and let the modified coupled BVP

$$-u_1''(t) + \lambda u_1(t) = \hat{F}_1\left(t, u_1(t), u_2(t), u_1'(t), u_2'(t)\right), \quad t \in [0,1],$$

$$-u_2''(t) + \lambda u_2(t) = \hat{F}_2\left(t, u_1(t), u_2(t), u_1'(t), u_2'(t)\right), \quad t \in [0,1],$$

$$\hat{\phi}\left(u_1(0), u_2(0), u_1(1), u_2(1), u_1'(0), u_2'(0)\right) = (u_1(0), u_2(0)),$$

$$\hat{\psi}\left(u_1(0), u_2(0), u_1(1), u_2(1), u_1'(1), u_2'(1)\right) = (u_1(1), u_2(1)), \tag{4.1.6}$$

with

$$\hat{F}_1\left(t, u_1(t), u_2(t), a(t), b(t)\right) = f_1\left(t, \tau_1(t, u_1(t)), \tau_2(t, u_2(t)), \sigma_1(a(t)), \sigma_1(b(t))\right) + \lambda\tau_1(t, u_1(t)),$$
$$\hat{F}_2\left(t, u_1(t), u_2(t), a(t), b(t)\right) = f_2\left(t, \tau_1(t, u_1(t)), \tau_2(t, u_2(t)), \sigma_1(a(t)), \sigma_2(b(t))\right) + \lambda\tau_2(t, u_2(t)),$$

$$\hat{\phi}(j,k,l,m,n,o) = (\tau_1(0,j), \tau_2(0,k)) + \phi\left(u_1(0), u_2(0), u_1(1), u_2(1), u_1'(0), u_2'(0)\right),$$
$$\hat{\psi}(j,k,l,m,n,o) = (\tau_1(1,l), \tau_2(1,m)) + \psi\left(u_1(0), u_2(0), u_1(1), u_2(1), u_1'(1), u_2'(1)\right),$$

$$\tau_i(t,x) = \begin{cases} \beta_i(t) & \text{if } x \not\leq \beta_i(t), \quad i = 1,2 \\ x & \text{if } \alpha_i(t) \leq x \leq \beta_i(t), \quad i = 1,2 \\ \alpha_i(t) & \text{if } x \not\geq \alpha_i(t), \quad i = 1,2 \end{cases}$$

and

$$\sigma_i(x) = \begin{cases} N_i & \text{if } x \not\leq N_i, \quad i = 1,2 \\ x & \text{if } -N_i \leq x \leq N_i, \quad i = 1,2 \\ -N_i & \text{if } x \not\geq -N_i, \quad i = 1,2. \end{cases}$$

Note that if $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$ is a solution of (4.1.6), then (u_1, u_2) is a solution of (4.1.1)-(4.1.2).

For the sake of simplicity we divide the proof in four steps:

Step 1: We define the mappings

$$L, N : C^2[0,1] \times C^2[0,1] \to C_0^2[0,1] \times C_0^2[0,1] \times \mathbb{R}^2 \times \mathbb{R}^2,$$

by

$$[L(u_1, u_2)](t) = \left(u_1'(t) - u_1'(0) - \lambda\int_0^t u_1(s)\,ds, u_2'(t) - u_2'(0) - \lambda\int_0^t u_2(s)\,ds,\right.$$

$$\left.(u_1(0), u_2(0)), (u_1(1), u_2(1))\right),$$

and

$$[N(u_1, u_2)](t) = \left(\int_0^t \hat{F}_1(s, u_1(s), u_2(s), u_1'(s), u_2'(s))ds, \int_0^t \hat{F}_2(s, u_1(s), u_2(s), u_1'(s), u_2'(s))ds,\right.$$

$$\phi(u_1(0), u_2(0), u_1(1), u_2(1), u_1'(0), u_2'(0)),$$

$$\left.\psi(u_1(0), u_2(0), u_1(1), u_2(1), u_1'(1), u_2'(1))\right).$$

Since $\hat{F}_1(s, u_1(s), u_2(s), u_1^{'}(s), u_2^{'}(s))$ and $\hat{F}_2(s, u_1(s), u_2(s), u_1^{'}(s), u_2^{'}(s))$ are bounded on $[0,1] \times \mathbb{R}^4$ and integral is a continuous function on $[0,1]$. Further $\hat{\phi}$ and $\hat{\psi}$ being constant functions are continuous. Therefore $[N(u_1, u_2)]$ is continuous on $[0,1]$. Further, the class $\{N(u_1, u_2) : u_1, u_2 \in C^2[0,1]\}$ is uniformly bounded and equicontinuous. Therefore in view of Arzelà-Ascoli theorem $\{N(u_1, u_2) : u_1, u_2 \in C^2[0,1]\}$ is relatively compact. Consequently N is a compact map. Also from Lemma 3.1.1 with $a = 1, b = 0, c = 1, d = 0$ and $E = 0, F = 1, G = 0, H = 1, L^{-1}$, exists and is continuous.

On the other hand, solving (4.1.6) is equivalent to find a fixed point of

$$L^{-1}N : C^2[0,1] \times C^1[0,1] \to C^2[0,1] \times C^1[0,1].$$

Now, Schauder's fixed point theorem guarantees the existence of at least a fixed point since $L^{-1}N$ is continuous and compact.

Step 2: If (u_1, u_2) is a solution of (4.1.6), then $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$. By definition of $\hat{\phi}$ and $\hat{\psi}$, it is obvious that, $(u_1(0), u_2(0)) \in [\alpha_1(0), \beta_1(0)] \times [\alpha_2(0), \beta_2(0)]$ and $(u_1(1), u_2(1)) \in [\alpha_1(1), \beta_1(1)] \times [\alpha_2(1), \beta_2(1)]$ respectively. We claim $(u_1, u_2) \preceq (\beta_1, \beta_2)$. If $(u_1, u_2) \npreceq (\beta_1, \beta_2)$, then $u_1 \npreceq \beta_1$ and/or $u_2 \npreceq \beta_2$. If $u_1 \npreceq \beta_1$, then $u_1 - \beta_1$ has positive maximum at some $r_0 \in (0,1)$ such that $(u_1 - \beta_1)^{'}(r_0) = 0$ and $(u_1 - \beta_1)^{''}(r_0) < 0$. But,

$$(u_1 - \beta_1)^{''}(r_0) > -\hat{F}\left(r_0, u_1(r_0), u_2(r_0), u_1^{'}(r_0), u_2^{'}(r_0)\right) + \lambda u_1(r_0) + f_1\left(r_0, \beta_1(r_0), \beta_2(r_0), \beta_1^{'}(r_0), \beta_2^{'}(r_0)\right)$$

$$= -f_1\left(r_0, \beta_1(r_0), \beta_2(r_0), u_1^{'}(r_0), u_2^{'}(r_0)\right) - \lambda\beta_1(r_0) + \lambda u_1(r_0)$$

$$+ f_1\left(r_0, \beta_1(r_0), \beta_2(r_0), \beta_1^{'}(r_0), \beta_2^{'}(r_0)\right)$$

$$= -f_1\left(r_0, \beta_1(r_0), \beta_2(r_0), \beta_1^{'}(r_0), \beta_2^{'}(r_0)\right) - \lambda\beta_1(r_0) + \lambda u_1(r_0)$$

$$+ f_1\left(r_0, \beta_1(r_0), \beta_2(r_0), \beta_1^{'}(r_0), \beta_2^{'}(r_0)\right)$$

$$= \lambda\left(u_1(r_0) - \beta_1(r_0)\right) > 0,$$

a contradiction. Similarly we can show that $(\alpha_1, \alpha_2) \preceq (u, v)$. Hence $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$.

Step 3: If (u_1, u_2) is a solution of (4.1.6), then (u_1, u_2) satisfies (4.1.2).

For this, we claim that

$$(\alpha_1(0), \alpha_2(0)) \preceq (u_1(0), u_2(0)) + \phi\left(u_1(0), u_2(0), u_1(1), u_2(1), u_1^{'}(0), u_2^{'}(0)\right) \preceq (\beta_1(0), \beta_2(0)). \quad (4.1.7)$$

If $(u_1(0), u_2(0)) + \phi\left(u_1(0), u_2(0), u_1(1), u_2(1), u_1^{'}(0), u_2^{'}(0)\right) \npreceq (\beta_1(0), \beta_2(0))$, then

$$(u_1(0), u_2(0)) = \hat{\phi}\left(u_1(0), u_2(0), u_1(1), u_2(1), u_1^{'}(0), u_2^{'}(0)\right)$$

$$= (\tau_1(0, u_1(0)), \tau_2(0, u_2(0))) + \phi(u_1(0), u_2(0), u_1(1), u_2(1), u_1^{'}(0), u_2^{'}(0)))$$

$$= (\beta_1(0), \beta_2(0)).$$

From Step 2, we know that $(u_1, u_2) \preceq (\beta_1, \beta_2)$, and this together with $(u_1 - \beta_1, u_2 - \beta_2) \in C^2[0,1] \times C^2[0,1]$ and $(u_1(0), u_2(0)) = (\beta_1(0), \beta_2(0))$ yields $u_1^{'}(0) \leq \beta_1^{'}(0)$ and

$u_2'(0) \leq \beta_2'(0)$. If $\phi_\beta(x,y)$ is monotone nonincreasing, then we have

$$
\begin{aligned}
(u_1(0), u_2(0)) \quad + \quad & \phi\left(u_1(0), u_2(0), u_1(1), u_2(1), u_1'(0), u_2'(0)\right) \\
= \quad & (\beta_1(0), \beta_2(0)) + \phi\left(\beta_1(0), \beta_2(0), u_1(1), u_2(1), u_1'(0), u_2'(0)\right) \\
\preceq \quad & (\beta_1(0), \beta_2(0)) + \phi\left(\beta_1(0), \beta_2(0), u_1(1), u_2(1), \beta_1'(0), \beta_2'(0)\right) \\
= \quad & (\beta_1(0), \beta_2(0)) + \phi_\beta\left(u_1(1), u_2(1)\right) \\
\preceq \quad & (\beta_1(0), \beta_2(0)) + \phi_\beta\left(\alpha_1(1), \alpha_2(1)\right) \\
= \quad & (\beta_1(0), \beta_2(0)) + \phi\left(\beta_1(0), \beta_2(0), \alpha_1(1), \alpha_2(1), \beta_1'(0), \beta_2'(0)\right) \\
\preceq \quad & (\beta_1(0), \beta_2(0)),
\end{aligned}
\tag{4.1.8}
$$

a contradiction. Similarly, if $\phi_\beta(x,y)$ is monotone nondecreasing, then we get same contradiction. Consequently, (4.1.7) holds. Similar reasoning shows the other boundary condition. Consequently, (u_1, u_2) satisfies (4.1.2). Hence the system of BVPs (4.1.1)-(4.1.2) has a solution $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$.

Step 4: If $(u_1, u_2) \in C^2[0,1] \times C^2[0,1]$ is a solution of (4.1.6) then $(-N_1, -N_2) \prec \left(u_1'(t), u_2'(t)\right) \prec (N_1, N_2)$. We claim $\left(u_1', u_2'\right) \prec (N_1, N_2)$. If $\left(u_1', u_2'\right) \not\prec (N_1, N_2)$, then $u_1' \not\prec N_1$ and/or $u_2' \not\prec N_2$. If $u_1' \not\prec N_1$, then there exists $s_0 \in [0,1]$ such that $u_1'(s_0) \geq N_1$. Moreover using the Lagrange theorem, there exists $t_0 \in [0,1]$ with $u_1'(t_0) = u_1(1) - u_1(0)$. So,

$$
-N_1 < -p_0 \leq \alpha_1(1) - \beta_1(0) + \alpha_2(1) - \beta_2(0) \leq u_1'(t_0) + u_2'(t_0) \leq \beta_1(0) - \alpha_1(0) + \beta_2(0) - \alpha_2(0) \leq p_0 < N_1.
$$

Now consider an interval $[s_1, s_2]$ or $[s_2, s_1]$ such that $u_1'(s_1) + u_2'(s_1) = p_0$ and $u_1'(s_2) + u_2'(s_2) = N_1$, either

$$
\begin{aligned}
p_0 = u_1'(s_1) + u_2'(s_1) \leq u_1'(t) + u_2'(t) \leq u_1'(s_2) + u_2'(s_2) = N_1, \quad t \in (s_1, s_2), \quad \text{or,} \\
p_0 = u_1'(s_1) + u_2'(s_1) \leq u_1'(t) + u_2'(t) \leq u_1'(s_2) + u_2'(s_2) = N_1, \quad t \in (s_2, s_1).
\end{aligned}
\tag{4.1.9}
$$

In the first situation we obtain from Definition 4.1.2 that

$$
\int_{u_1'(s_1)+u_2'(s_1)}^{u_1'(s_2)+u_2'(s_2)} \frac{dy}{g(y)} = \int_{p_0}^{N_1} \frac{dy}{g(y)} > 2.
$$

Using (4.1.4), Step 2 and $N_1 \geq u_1'(t) + u_2'(t) \geq p_0 \geq 0$, for all $t \in (s_1, s_2)$, we get the contradiction

$$
\begin{aligned}
\int_{u_1'(s_1)+u_2'(s_1)}^{u_1'(s_2)+u_2'(s_2)} \frac{dy}{g(y)} \quad = \quad & \int_{s_1}^{s_2} \frac{u_1''(t) + u_2''(t)}{g(u_1'(t) + u_2'(t))} dt \\
= \quad & \int_{s_1}^{s_2} \frac{u_1''(t)}{g(u_1'(t) + u_2'(t))} dt + \int_{s_1}^{s_2} \frac{u_2''(t)}{g(u_1'(t) + u_2'(t))} dt \\
= \quad & \int_{s_1}^{s_2} \frac{-\hat{F}_1\left(t, u_1(t), u_2(t), u_1'(t), u_2'(t)\right) + \lambda u_1(t)}{g(u_1'(t) + u_2'(t))} \\
& + \int_{s_1}^{s_2} \frac{-\hat{F}_2\left(t, u_1(t), u_2(t), u_1'(t), u_2'(t)\right) + \lambda u_2(t)}{g(u_1'(t) + u_2'(t))} \\
= \quad & \int_{s_1}^{s_2} \frac{-f_1\left(t, u_1(t), u_2(t), u_1'(t), u_2'(t)\right)}{g(u_1'(t) + u_2'(t))} dt + \int_{s_1}^{s_2} \frac{-f_2\left(t, u_1(t), u_2(t), u_1'(t), u_2'(t)\right)}{g(u_1'(t) + u_2'(t))} dt
\end{aligned}
$$

$$\leq \int_{s_1}^{s_2} \frac{|f_1\left(t, u_1(t), u_2(t), u_1^{'}(t), u_2^{'}(t)\right)|}{g(u_1^{'}(t) + u_2^{'}(t))} dt + \int_{s_1}^{s_2} \frac{|f_2\left(t, u_1(t), u_2(t), u_1^{'}(t), u_2^{'}(t)\right)|}{g(u_1^{'}(t) + u_2^{'}(t))} dt$$

$$\leq \int_{s_1}^{s_2} \frac{g(u_1^{'}(t) + u_2^{'}(t))}{g(u_1^{'}(t) + u_2^{'}(t))} dt + \int_{s_1}^{s_2} \frac{g(u_1^{'}(t) + u_2^{'}(t))}{g(u_1^{'}(t) + u_2^{'}(t))} dt$$

$$= \int_{s_1}^{s_2} dt + \int_{s_1}^{s_2} dt$$

$$= 2(s_2 - s_1) \leq 2.$$

Similarly in the second situation we get a contradiction. Hence $\left(u_1^{'}(t), u_2^{'}(t)\right) \prec (N_1, N_2)$. The proof of other inequality is similar. $\qquad \square$

4.1.5 Application of the theoretical results

In this section, we discuss the practical applicability of our theoretical results with the help of a example.

Example 4.1.1. *Consider the nonlinear coupled system of BVPs*

$$-u_1^{''}(t) = -\left(t^2 + 3\right)\left(e^{u_1(t)} + e^{u_2(t)}\right) - 2\left((u_1^{'})^2 + (u_2^{'})^2\right) + 2\left(t^2 + 1\right), \quad t \in [0, 1],$$

$$-u_2^{''}(t) = -\left(t^4 + 9\right)\left(e^{2u_1(t)} + e^{2u_2(t)}\right) - 4\left((u_1^{'})^4 + (u_2^{'})^4\right) + 4\left(t^4 + 2\right), \quad t \in [0, 1], \quad (4.1.10)$$

subject to nonlinear CBCs

$$\left(u_1(0)\sin(u_2(0)) + \sin\left(u_1^{'}(0)\right)u_2(0), u_1(0)\sin\left(u_2^{'}(0)\right) + u_1^{'}(0)u_2^{'}(0)\right) = (0, 0),$$

$$\left(u_1(0)\tan(u_2(0)) - \sin\left(u_1^{'}(1)\right)u_2^{'}(1), -u_1^{'}(1)\tan\left(u_2^{'}(1)\right) + u_2(0)u_1(1)\right) = (0, 0). \quad (4.1.11)$$

The nonlinearities $f_1(t, x_0, x_1, x_2, x_3) = -\left(t^2 + 3\right)(e^{x_0} + e^{x_1}) - 2\left((x_2)^2 + (x_3)^2\right) + 2\left(t^2 + 1\right)$ and $f_2(t, x_0, x_1, x_2, x_3) = -\left(t^4 + 9\right)\left(e^{2x_0} + e^{2x_1}\right) - 4\left((x_2)^4 + (x_3)^4\right) + 4\left(t^4 + 2\right)$ are continuous on $[0, 1] \times \mathbb{R}^4$. Then the functions $(\alpha_1, \alpha_2), (\beta_1, \beta_2) : [0, 1] \times [0, 1] \to \mathbb{R} \times \mathbb{R}$ defined by
  $\alpha_1(t) = -2, \ \alpha_2(t) = -4, \ \beta_1(t) = t^2, \ and \ \beta_2(t) = t^4$
are, respectively, lower and upper solutions of system (4.1.10), according to Definition 4.1.1.
  Moreover, it can be easily check that the functions $\alpha_1(t) = -2, \ \alpha_2(t) = -4, \ \beta_1(t) = t^2, and \ \beta_2(t) = t^4$ satisfy system of inequalities (4.1.5). Consequently, these functions are, respectively, coupled lower and upper solutions of (4.1.10)-(4.1.11), according to Definition 4.1.3.
  Also notice that the functions $f_1(t, x_0, x_1, x_2, x_3) = -\left(t^2 + 3\right)(e^{x_0} + e^{x_1}) - 2\left((x_2)^2 + (x_3)^2\right) + 2\left(t^2 + 1\right)$ and $f_2(t, x_0, x_1, x_2, x_3) = -\left(t^4 + 9\right)\left(e^{2x_0} + e^{2x_1}\right) - 4\left((x_2)^4 + (x_3)^4\right) + 4\left(t^4 + 2\right)$ satisfies the Nagumo condition (4.1.4) with $g(x_2 + x_3) = 4(1 - 2e) - 2\left((x_2)^2 + (x_3)^2\right), x_2, x_3 \in \mathbb{R}$ and $g(x_2 + x_3) = 4\left(3 - 5e^2 - (x_2)^4 - (x_3)^4\right)$ with $x_2, x_3 \in \mathbb{R}$, respectively.
  Hence, by Theorem 4.1.1, there is at least one solution $(u_1(t), u_2(t))$ of the BVPs (4.1.10)-(4.1.11) such that, for every $t \in [0, 1]$,

$$(-2, -4) \preceq (u_1(t), u_2(t)) \preceq \left(t^2, t^4\right).$$

4.2 Second-order coupled system with derivative dependent nonlinearity (II)

In this section, we investigate the existence of solutions of nonlinear coupled system of BVPs of the type

$$-u_1^{''}(t) = f_1(t, u_1(t), u_2(t), u_1^{'}(t), u_2^{'}(t)), \quad t \in [0, 1],$$
$$-u_2^{''}(t) = f_2(t, u_1(t), u_2(t), u_1^{'}(t), u_2^{'}(t)), \quad t \in [0, 1], \tag{4.2.1}$$

$$\mu(u_1(0), u_2(0), u_1^{'}(0), u_2^{'}(0), u_1^{'}(1), u_2^{'}(1)) = (0, 0),$$
$$\nu(u_1(0), u_2(0)) + (u_1(1), u_2(1)) = (0, 0), \tag{4.2.2}$$

where $f_1, f_2 : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\mu : \mathbb{R}^6 \to \mathbb{R}^2$ and $\nu : \mathbb{R}^2 \to \mathbb{R}^2$ are continuous functions.

We establish the existence of at-least one solution of (4.2.1)–(4.2.2) with the help of LUSs method in the presence of the existence of well ordered LUSs such that $(\alpha_1, \alpha_2) \preceq (\beta_1, \beta_2)$ on $[0, 1] \times [0, 1]$.

The monotonicity assumptions on ϕ and ψ as stated in Theorem 4.1.1 does not address to the existence criterion for a very important BVPs like periodic and anti-periodic. But in this study, we introduce a new generalized BCs 4.2.2 that satisfy the existence criterion for a periodic and anti-periodic BVPs as a particular case under some monotonicity assumptions on the arguments of the boundary functions μ and ν as discussed in detail in the introductory chapter of this thesis.

This section deals to the concept of CLUSs that unifies the treatment of many linear and nonlinear BVPs under some monotonicity assumptions on the arguments of the boundary functions μ and ν.

4.2.1 Coupled lower and upper solutions

Definition 4.2.1. We say that a couple of functions (α_1, α_2), $(\beta_1, \beta_2) \in C^2[0, 1] \times C^2[0, 1]$ are coupled lower and upper solutions for the coupled BVP (4.2.1) and (4.2.2), if (α_1, α_2) is a lower solution and (β_1, β_2) is an upper solution for the system (4.2.1),$(\alpha_1, \alpha_2) \preceq (\beta_1, \beta_2)$, if $\alpha_1 \leq \beta_1$ and $\alpha_2 \leq \beta_2$, and

$$\mu(\beta_1(0), \beta_2(0), \beta_1^{'}(0), \beta_2^{'}(0), \beta_1^{'}(1), \beta_2^{'}(1)) \preceq (0, 0) \preceq \mu(\alpha_1(0), \alpha_2(0), \alpha_1^{'}(0), \alpha_2^{'}(0), \alpha_1^{'}(1), \alpha_2^{'}(1)),$$
$$\mu(\beta_1(0), \beta_2(0), \beta_1^{'}(0), \beta_2^{'}(0), \alpha_1^{'}(1), \alpha_2^{'}(1)) \preceq (0, 0) \preceq \mu(\alpha_1(0), \alpha_2(0), \alpha_1^{'}(0), \alpha_2^{'}(0), \beta_1^{'}(1), \beta_2^{'}(1)),$$
$$(\alpha_1(1), \alpha_2(1)) + \nu(\beta_1(0), \beta_2(0)) = (0, 0),$$
$$(\beta_1(1), \beta_2(1)) + \nu(\alpha_1(0), \alpha_2(0)) = (0, 0),$$
$$(\alpha_1(1), \alpha_2(1)) + \nu(\alpha_1(0), \alpha_2(0)) = (0, 0),$$
$$(\beta_1(1), \beta_2(1)) + \nu(\beta_1(0), \beta_2(0)) = (0, 0). \tag{4.2.3}$$

4.2.2 Existence of at least one solution

Theorem 4.2.1. *Suppose the nonlinear coupled BVP* (4.2.1)-(4.2.2) *having* (α_1, α_2) *and* (β_1, β_2) *coupled lower and upper solutions respectively and the nonlinear functions* f_1 *and* f_2 *relative to the intervals* $[\alpha_1(t), \beta_1(t)]$ *and* $[\alpha_2(t), \beta_2(t)]$ *satisfy the Nagumo assumptions respectively. Moreover, assume that* μ *is nondecreasing in the third and fourth arguments. In addition suppose that the function* ν *in* $[\alpha_1(0), \beta_1(0)] \times [\alpha_2(0), \beta_2(0)]$ *is monotone and the functions*

$$\mu_{(\alpha_1, \alpha_2)}(x, y) := \mu\left(\alpha_1(0), \alpha_2(0), \alpha_1'(0), \alpha_2'(0), x, y\right),$$

$$\mu_{(\beta_1, \beta_2)}(x, y) := \mu\left(\beta_1(0), \beta_2(0), \beta_1'(0), \beta_2'(0), x, y\right),$$

have got the same kind of monotonicity as ν, *Then the nonlinear coupled system of BVPs* (4.2.1)-(4.2.2) *have at least one solution* $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$ *and* $(-N_1, -N_2) \preceq (u_1'(t), u_2'(t)) \preceq (N_1, N_2), t \in [0, 1]$.

Proof. Let $\lambda > 0$, and consider the modified system of BVP

$$-u_1''(t) + \lambda u_1(t) = F_1^*\left(t, u_1(t), u_2(t), u_1'(t), u_2'(t)\right), \quad t \in [0, 1],$$

$$-u_2''(t) + \lambda u_2(t) = F_2^*\left(t, u_1(t), u_2(t), u_1'(t), u_2'(t)\right), \quad t \in [0, 1],$$

$$\mu^*\left(u_1(0), u_2(0), u_1(1), u_2(1), u_1'(0), u_2'(0)\right) = (u_1(0), u_2(0)),$$

$$(u_1(1), u_2(1)) + \nu^*\left(u_1(0), u_2(0)\right) = (0, 0), \tag{4.2.4}$$

with

$$F_1^*\left(t, u_1(t), u_2(t), r(t), s(t)\right) = f_1\left(t, \Lambda_1(t, u_1(t)), \Lambda_2(t, u_2(t)), \Upsilon_1(r(t)), \Upsilon_2(s(t))\right) + \lambda \Lambda_1(t, u_1(t)),$$

$$F_2^*\left(t, u_1(t), u_2(t), r(t), s(t)\right) = f_2\left(t, \Lambda_1(t, u_1(t)), \Lambda_2(t, u_2(t)), \Upsilon_1(r(t)), \Upsilon_2(s(t))\right) + \lambda \Lambda_2(t, u_2(t)),$$

$$\mu^*(j, k, l, m, n, o) = (\Lambda_1(0, j), \Lambda_2(0, k)) + \mu(j, k, l, m, n, o),$$

$$\nu^*(j, k) = \nu(\Lambda_1(0, j), \Lambda_2(0, k)),$$

$$\Lambda_i(t, z) = \begin{cases} \beta_i(t) & \text{if } z \nleq \beta_i(t), \quad i = 1, 2, \\ z & \text{if } \alpha_i(t) \leq z \leq \beta_i(t), \quad i = 1, 2, \\ \alpha_i(t) & \text{if } z \ngeq \alpha_i(t), \quad i = 1, 2, \end{cases}$$

and

$$\Upsilon_i(z) = \begin{cases} N_i & \text{if } z \nleq N_i, \quad i = 1, 2, \\ z & \text{if } -N_i \leq z \leq N_i, \quad i = 1, 2, \\ -N_i & \text{if } z \ngeq -N_i, \quad i = 1, 2. \end{cases}$$

Note that if $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$ is a solution of (4.2.4), then (u_1, u_2) is a solution of (4.2.1)-(4.2.2).

For the sake of simplicity we divide the proof in four steps:

Step 1: We define the mappings

$$L, N : C^2[0, 1] \times C^2[0, 1] \to C_0^2[0, 1] \times C_0^2[0, 1] \times \mathbb{R}^2 \times \mathbb{R}^2,$$

by

$$[L(u_1, u_2)](t) = \left(u_1^{'}(t) - u_1^{'}(0) - \lambda \int_0^t u_1(s)\,\mathrm{d}s, u_2^{'}(t) - u_2^{'}(0) - \lambda \int_0^t u_2(s)\,\mathrm{d}s, \right.$$
$$\left. (u_1(0), u_2(0)), (u_1(1), u_2(1)) \right),$$

and

$$[N(u_1, u_2)](t) = \left(\int_0^t F_1^*(s, u_1(s), u_2(s), u_1^{'}(s), u_2^{'}(s))ds, \int_0^t F_2^*(s, u_1(s), u_2(s), u_1^{'}(s), u_2^{'}(s))ds, \right.$$
$$\mu^*(u_1(0), u_2(0), u_1(1), u_2(1), u_1^{'}(0), u_2^{'}(0)),$$
$$\left. -\nu^*(u_1(0), u_2(0)) \right).$$

Since $F_1^*(s, u_1(s), u_2(s), u_1^{'}(s), u_2^{'}(s))$ and $F_2^*(s, u_1(s), u_2(s), u_1^{'}(s), u_2^{'}(s))$ are bounded on $[0,1] \times \mathbb{R}^4$ and integral is a continuous function on $[0,1]$. Further μ^* and ν^* being constant functions are continuous. Therefore $[N(u_1, u_2)]$ is continuous on $[0,1]$. Further, the class $\{N(u_1, u_2) : u_1, u_2 \in C^2[0,1]\}$ is uniformly bounded and equicontinuous. Therefore in view of Arzelà-Ascoli theorem $\{N(u_1, u_2) : u_1, u_2 \in C^2[0,1]\}$ is relatively compact. Consequently N is a compact map. Also from Lemma 3.1.1 with $a = 1, b = 0,\ c = 1, d = 0$ and $E = 0,\ F = 1,\ G = 0,\ H = 1,\ L^{-1}$, exists and is continuous.

On the other hand, solving (4.2.4) is equivalent to find a fixed point of

$$L^{-1}N : C^2[0,1] \times C^2[0,1] \to C^2[0,1] \times C^2[0,1].$$

Now, Schauder's fixed point theorem guarantees the existence of at least a fixed point since $L^{-1}N$ is continuous and compact.

Step 2: If (u_1, u_2) is a solution of (4.2.4), then $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$. By definition of μ^*, we see that $(u_1(0), u_2(0)) \in [\alpha_1(0), \beta_1(0)] \times [\alpha_2(0), \beta_2(0)]$. Thus, if ν is nondecreasing, we have by condition (4.2.3)

$$(\alpha_1(1), \alpha_2(1)) = -\nu(\beta_1(0), \beta_2(0)) \preceq -\nu(u_1(0), u_2(0)) = (u_1(1), u_2(1)) \preceq -\nu(\alpha_1(0), \alpha_2(0))$$
$$(\alpha_1(1), \alpha_2(1)) \preceq (u_1(1), u_2(1)) \preceq (\beta_1(0), \beta_2(0)), \tag{4.2.5}$$

similarly, if ν is non-increasing, then (4.2.5) holds. Hence $(u_1(1), u_2(1)) \in [\alpha_1(1), \beta_1(1)] \times [\alpha_2(1), \beta_2(1)]$. Now, it remains to show that $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$ for $t \in (0,1)$.

We claim $(u_1, u_2) \preceq (\beta_1, \beta_2)$. If $(u_1, u_2) \not\preceq (\beta_1, \beta_2)$, then either $u_1 \not\preceq \beta_1$ and/or $u_2 \not\preceq \beta_2$. If $u_1 \not\preceq \beta_1$, then there exists some $t_0 \in [0,1]$ such that $u_1(t_0) - \beta_1(t_0) > 0$. So, $u_1 - \beta_1$ attains a positive maximum at $t_0 \in [0,1]$. Thus $(u_1 - \beta_1)^{'}(t_0) = 0$ and $(u_1 - \beta_1)^{''}(t_0) < 0$. But,

$$(u_1 - \beta_1)^{''}(t_0) > -F_1^*\left(t_0, u_1(t_0), u_2(t_0), u_1^{'}(t_0), u_2^{'}(t_0) \right) + \lambda u_1(t_0) + f_1\left(t_0, \beta_1(t_0), \beta_2(t_0), \beta_1^{'}(t_0), \beta_2^{'}(t_0) \right)$$
$$= -f_1\left(t_0, \beta_1(t_0), \beta_2(t_0), u_1^{'}(t_0), u_2^{'}(t_0) \right) - \lambda \beta_1(t_0) + \lambda u_1(t_0)$$
$$+ f_1\left(t_0, \beta_1(t_0), \beta_2(t_0), \beta_1^{'}(t_0), \beta_2^{'}(t_0) \right)$$
$$= -f_1\left(t_0, \beta_1(t_0), \beta_2(t_0), \beta_1^{'}(t_0), \beta_2^{'}(t_0) \right) - \lambda \beta_1(t_0) + \lambda u_1(t_0)$$
$$+ f_1\left(t_0, \beta_1(t_0), \beta_2(t_0), \beta_1^{'}(t_0), \beta_2^{'}(t_0) \right)$$
$$= \lambda(u_1(t_0) - \beta_1(t_0)) > 0,$$

a contradiction. Similarly one can show that $(\alpha_1, \alpha_2) \preceq (u_1, u_2)$. Hence $(u_1, u_2) \in [\![\alpha_1, \beta_1]\!] \times [\![\alpha_2, \beta_2]\!]$.
Step 3: If (u_1, u_2) is a solution of (4.2.4) then (u_1, u_2) satisfies (4.2.2).

We claim

$$(\alpha_1(0), \alpha_2(0)) \preceq (u_1(0), u_2(0)) + \mu \left(u_1(0), u_2(0), u_1^{'}(0), u_2^{'}(0), u_1^{'}(1), u_2^{'}(1) \right) \preceq (\beta_1(0), \beta_2(0)) . \quad (4.2.6)$$

If $(\alpha_1(0), \alpha_2(0)) \npreceq (u_1(0), u_2(0)) + \mu \left(u_1(0), u_2(0), u_1^{'}(0), u_2^{'}(0), u_1^{'}(1), u_2^{'}(1) \right)$, then

$$\begin{aligned}
(u_1(0), u_2(0)) &= \mu^* \left(u_1(0), u_2(0), u_1^{'}(0), u_2^{'}(0), u_1^{'}(1), u_2^{'}(1) \right) \\
&= (\Lambda_1(0, u_1(0)), \Lambda_2(0, u_2(0))) + \mu(u_1(0), u_2(0), u_1^{'}(0), u_2^{'}(0), u_1^{'}(1), u_2^{'}(1))) \\
&= (\alpha_1(0), \alpha_2(0)) . \quad (4.2.7)
\end{aligned}$$

If ν is non-decreasing then we have

$$(u_1(1), u_2(1)) = -\nu (u_1(0), u_2(0)) = -\nu (\alpha_1(0), \alpha_2(0)) = (\beta_1(1), \beta_2(1)) . \quad (4.2.8)$$

Using (4.2.7),(4.2.8) and Step 2, we have $\left(u_1^{'}(0), u_2^{'}(0) \right) \succeq \left(\alpha_1^{'}(0), \alpha_2^{'}(0) \right)$ and $\left(u_1^{'}(1), u_2^{'}(1) \right) \succeq \left(\beta_1^{'}(1), \beta_2^{'}(1) \right)$. But

$$\begin{aligned}
(u_1(0), u_2(0)) \quad &+ \quad \mu \left(u_1(0), u_2(0), u_1^{'}(0), u_2^{'}(0), u_1^{'}(1), u_2^{'}(1) \right) \\
&= \quad (\alpha_1(0), \alpha_2(0)) + \mu \left(\alpha_1(0), \alpha_2(0), u_1^{'}(0), u_2^{'}(0), u_1^{'}(1), u_2^{'}(1) \right) \\
&\succeq \quad (\alpha_1(0), \alpha_2(0)) + \mu \left(\alpha_1(0), \alpha_2(0), \alpha_1^{'}(0), \alpha_2^{'}(0), u_1^{'}(1), u_2^{'}(1) \right) \\
&= \quad (\alpha_1(0), \alpha_2(0)) + \mu_{(\alpha_1, \alpha_2)} \left(u_1^{'}(1), u_2^{'}(1) \right) \\
&\succeq \quad (\alpha_1(0), \alpha_2(0)) + \mu \left(\alpha_1(0), \alpha_2(0), \alpha_1^{'}(0), \alpha_2^{'}(0), \beta_1^{'}(1), \beta_2^{'}(1) \right) \\
&\succeq \quad (\alpha_1(0), \alpha_2(0)) , \quad (4.2.9)
\end{aligned}$$

a contradiction. Similarly if ν is non-increasing we get same contradiction. Consequently, (4.2.6) holds. By definition of ν^* and Step 2, the second boundary condition is obvious. Consequently (u_1, u_2) satisfies (4.2.2).
Now the rest of the proof is analogous to the proof of the Theorem 4.1.1. □

4.2.3 Application of the theoretical results

In this section we check the validity of the Theorem 4.2.1 by considering an example.

Example 4.2.1. *Consider the following nonlinear coupled system of BVP*

$$-u_1^{''}(t) = -5 \left(u_1^{'}(t) + u_2^{'}(t) \right) - (u_1(t) + 1)^2 - (u_2(t) + 1)^2 + \sin^2(\pi t), \quad t \in [0, 1],$$

$$-u_2^{''}(t) = -10 \left(u_1^{'}(t) + 2u_2^{'}(t) \right) - (u_1(t) + 2)^4 - (u_2(t) + 1)^4 + \sin^2(\pi t), \quad t \in [0, 1], \quad (4.2.10)$$

corresponding to nonlinear CBCs

$$\left(u_1(0)u_1^{'}(0) - u_2(0)u_2^{'}(0), u_1(0)u_1^{'}(1) - u_2(0)u_2^{'}(1) \right) = (0, 0) ,$$

$$(u_1(0)u_2(0) + u_1(0)u_2(1), u_1(1)u_2(0) + u_1(0)u_2(1)) = (0, 0) . \quad (4.2.11)$$

The nonlinearities $f_1(t, x_0, x_1, x_2, x_3) = -5\left(x_2 + x_3\right) - \left(x_0 + 1\right)^2 - \left(x_1 + 1\right)^2 + \sin^2(\pi t)$ and $f_2(t, x_0, x_1, x_2, x_3) = -10\left(x_2 + 2x_3\right) - \left(x_0 + 2\right)^4 - \left(x_1 + 1\right)^4 + \sin^2(\pi t)$ are continuous on $[0, 1] \times \mathbb{R}^4$. Then the functions $\left(\alpha_1, \alpha_2\right), \left(\beta_1, \beta_2\right) : [0, 1] \times [0, 1] \to \mathbb{R} \times \mathbb{R}$ defined by

$\alpha_1(t) = -t^2 - t, \ \alpha_2(t) = -t \ \ and \ \ \beta_1(t) = t^2 + t, \beta_2(t) = t$

are, respectively, lower and upper solutions of the coupled nonlinear system (4.2.10), according to the Definition 4.1.1.

Moreover, it can be easily check that the functions $\alpha_1(t) = -t^2 - t, \ \alpha_2(t) = -t \ \ and \ \ \beta_1(t) = t^2 + t, \beta_2(t) = t$ satisfy (4.2.3). Consequently, these functions are, respectively, coupled lower and upper solutions of the coupled BVP (4.2.10)–(4.2.11), according to the Definition 4.2.1.

Also notice that the functions $f_1(t, x_0, x_1, x_2, x_3) = -5\left(x_2 + x_3\right) - \left(x_0 + 1\right)^2 - \left(x_1 + 1\right)^2 + \sin^2(\pi t)$ and $f_2(t, x_0, x_1, x_2, x_3) = -10\left(x_2 + 2x_3\right) - \left(x_0 + 2\right)^4 - \left(x_1 + 1\right)^4 + \sin^2(\pi t)$ satisfies the Nagumo condition (4.1.4) with $g(x_2 + x_3) = -5(x_2 + x_3) - 1, \ \ x_2, x_3 \in \mathbb{R}$ and $g(x_2 + x_3) = -10(x_2 + x_3), \ \ x_2, x_3 \in \mathbb{R}$, respectively.

Hence, by Theorem 4.2.1, there is at least one solution $(u_1(t), u_2(t))$ of the coupled BVP (4.2.10)-(4.2.11) such that, for all $t \in [0, 1]$,

$$\left(-t^2 - t, -t\right) \preceq (u_1(t), u_2(t)) \preceq \left(t^2 + t, t\right).$$

.

Chapter 5

Coupled Systems Of Multi-terms Fractional Order Partial Differential Equations

Partial differential equations of fractional order have rich applications in almost every field of science based including, but not limited to, fluid dynamics, physics, chemistry, aerodynamics, signal and image processing, chemical engineering, economics, and even in psychology, [9, 81, 84, 101, 142, 161, 176, 177].

The applications of coupled systems of fractional order partial differential equations (FOPDEs) have been widely discussed in the existing literature. For example in Biomechanics, coupled systems are used to model the electrical activity of the heart [168, 169], in chemical and material engineering they are used to model the systems consisting of a plug flow reactor and a continuous stirred tank reactor [10, 135], in solid mechanics, the physical phenomenon used to discuss the dynamics of multi-deformable bodies are modeled using coupled systems of FOPDEs [112, 152].

Based on a preliminary mathematical modeling, determination of exact and/or approximate solutions of fractional order differential equations has proved to be a major emerging research area that has captured the interest of the researchers round the world [35, 36, 58, 92, 93, 94, 100, 127, 136, 155, 187]. For instance, many scientists have devoted themselves to establishing efficient and reliable numerical schemes for finding the approximate solutions of FOPDEs using orthogonal polynomials which reduce them into an easily tractable system of algebraic equations.

Certain types of FOPDES where the non-integer order is left arbitrary, have been solved numerically applying such ideas and types of orthogonal polynomials, [23, 51, 113, 114, 115]. Recently, particular types of FOPDES have been solved by constructing new operational matrices of fractional derivatives and integrals based on such schemes using both orthogonal and non-orthogonal polynomials, [57, 115, 155, 187]. In [113, 114], the numerical schemes are developed with the help of operational matrices based on orthogonal Legendre polynomials.

Based on above works, we opted to develop an efficient numerical scheme able to treat a generalized class of coupled systems of FOPDEs, by developing the operational matrices for a particular kind of polynomials, namely, the Jacobi polynomials, (JPs). Unlike in the case of the Legendre polynomials, the JPs have the ability to approximate the solutions with the aid of two parameters, which is indeed a more generalized way of approximation, since in comparison to the works, [113, 114], herein, the operational matrices developed can deal with the FOPDEs having mixed partial derivatives of fractional order. Moreover, in [115], the operational matrices has been developed using one dimensional JPs. In our case, operational matrices are exhibited for two dimensional JPs.

This chapter is organized as: In Section 5.1.1, the two dimensional orthogonal shifted JPs are presented along-with their orthogonality expressions. In Section 5.1.2, we develop the analytical relation to determine the error bound for a sufficiently smooth function. In Section 5.1.3, the operational matrices of fractional order integrals and derivatives are established. In Section 5.2, the applications of the developed matrices of fractional order integrals and derivatives are discussed by developing the new schemes for the multi-terms FOPDEs and its coupled systems. In Section 5.2.3, the applicability of the developed method is checked by considering some test examples. In Section 5.3.1, the notion of three-dimensional orthogonal shifted JPs is discussed along-with their orthogonality relationship. In Section 5.3.3, the analytical relation is developed to determine the error bound for a sufficiently smooth function. In Section 5.3.4, the applications of fractional order integrals and derivatives is discussed by developing the numerical scheme for finding the approximate solutions of high dimensional multi-terms FOPDEs. In Section 5.3.5, some illustrative examples are tested to check the reliability and efficiency of the proposed method.

5.1 Multi-term fractional partial differential equations mixed type Coupled Systems

In this section, we are interested in developing the numerical scheme based on operational matrices of fractional order integrals and derivatives of the following multi-term FOPDEs and its coupled systems of the type

$$\frac{\partial^{\sigma_1} U(x,y)}{\partial x^{\sigma_1}} = a_1 \frac{\partial^{\gamma_1} U(x,y)}{\partial y^{\gamma_1}} + a_2 \frac{\partial^{\varrho_1} U(x,y)}{\partial x^{\varrho_1/2} \partial y^{\varrho_1/2}} + G_1(x,y), \tag{5.1.1}$$

corresponding to the following initial conditions with $j = 0, 1, ..., n$

$$U^{(j)}(0,y) = H_j(y), \tag{5.1.2}$$

and a coupled system of a generalized class of FOPDEs

$$\begin{aligned}
\frac{\partial^{\sigma_1} U(x,y)}{\partial x^{\sigma_1}} =& a_1 \frac{\partial^{\gamma_1} U(x,y)}{\partial y^{\gamma_1}} + a_2 \frac{\partial^{\gamma_2} V(x,y)}{\partial x^{\gamma_2}} + a_3 \frac{\partial^{\gamma_3} V(x,y)}{\partial y^{\gamma_3}} + \\
& a_4 \frac{\partial^{\varrho_4} U(x,y)}{\partial x^{\varrho_4/2} \partial y^{\varrho_4/2}} + a_5 \frac{\partial^{\varrho_3} V(x,y)}{\partial x^{\varrho_3/2} \partial y^{\varrho_3/2}} + G_1(x,y), \\
\frac{\partial^{\sigma_2} V(x,y)}{\partial x^{\sigma_2}} =& b_1 \frac{\partial^{\rho_1} V(x,y)}{\partial y^{\rho_1}} + b_2 \frac{\partial^{\rho_2} U(x,y)}{\partial x^{\rho_2}} + b_3 \frac{\partial^{\rho_3} U(x,y)}{\partial y^{\rho_1}} + \\
& b_4 \frac{\partial^{\varrho_1} U(x,y)}{\partial x^{\varrho_1/2} \partial y^{\varrho_1/2}} + b_5 \frac{\partial^{\varrho_2} V(x,y)}{\partial x^{\varrho_2/2} \partial y^{\varrho_2/2}} + G_2(x,y),
\end{aligned} \tag{5.1.3}$$

corresponding to the following initial conditions with $j = 0, 1, ..., n$

$$U^{(j)}(0,y) = H_j(y), \ V^{(j)}(0,y) = G_j(y), \tag{5.1.4}$$

where $n < \sigma_1, \sigma_2 \leq n+1$, a_j, b_j, $j = 1, ..., 5$ are all real constants and $F_1(x,y), F_2(x,y), U(x,y), V(x,y), \in C([0,\Delta] \times [0,\Delta])$.

We recall the concept of two-dimensional shifted JPs in two parameters.

5.1.1 Two parametric shifted Jacobi polynomials in two dimensions

This section elaborates the idea of orthogonal shifted JPs in two dimensions along-with its orthogonality relationship with the aid of a weight function. The following relation defines the one–dimensional shifted orthogonal JPs on the interval $[0, \Delta]$ as, (see [115])

$$Q_{\Delta,j}^{(\tau_1,\tau_2)}(x) = \sum_{l=0}^{j} \frac{(-1)^{j-l} \Gamma(j + \tau_2 + 1) \Gamma(j + l + \tau_1 + \tau_2 + 1)}{\Gamma(l + \tau_2 + 1) \Gamma(j + \tau_1 + \tau_2 + 1)(j - l)! l! \Delta^l} x^l, \quad j = 1, 2, 3, ... \tag{5.1.5}$$

The orthogonality expression is

$$\int_0^1 Q_{\Delta,j}^{(\tau_1,\tau_2)}(x) Q_{\Delta,i}^{(\tau_1,\tau_2)}(x) R_{\Delta}^{(\tau_1,\tau_2)}(x) dx = W_{\Delta,i}^{(\tau_1,\tau_2)} \delta_{j,i}, \tag{5.1.6}$$

where weight function $R_{\Delta}^{(\tau_1,\tau_2)}$ is defined as follows

$$R_{\Delta}^{(\tau_1,\tau_2)}(x) = (\Delta - x)^{\tau_1} x^{\tau_2}, \tag{5.1.7}$$

and

$$W_{\Delta,i}^{(\tau_1,\tau_2)} \delta_{j,i} = \frac{\Delta^{\tau_1+\tau_2+1} \Gamma(i+\tau_1+1) \Gamma(i+\tau_2+1)}{(2i+\tau_1+\tau_2+1) \Gamma(i+1) \Gamma(i+\tau_1+\tau_2+1)}. \tag{5.1.8}$$

It means that any integrable function $r(x)$ defined on $[0, \Delta]$ can be approximated in terms of the series of orthogonal shifted JPs as follows

$$r(x) \approx \sum_{b=0}^{n} D_b Q_{\Delta,b}^{(\tau_1,\tau_2)}(x), \tag{5.1.9}$$

The series approximation approaches to the exact function as $n \to \infty$. With the help of (5.1.6), (5.1.7) and (5.1.8), the coefficient D_b can be easily computed. The vector expression of (5.1.9) is as following

$$r(x) \approx G_N^T \hat{\Phi}_N(x), \tag{5.1.10}$$

where $\hat{\Phi}_N(x)$ represents the N terms vector function with $N = n+1$, and G_N^T indicates the coefficient vector. On the same fashion the two–dimension JPs of order N defined on $[0, \Delta] \times [0, \Delta]$ can be described using the following relation

$$Q_{\Delta,m}^{(\tau_1,\tau_2)}(x,y) = (Q_{\Delta,j}^{(\tau_1,\tau_2)}(x))(Q_{\Delta,i}^{(\tau_1,\tau_2)}(y)), \; m = Nj+i+1, \; j = 0,1,2,...,n, \; i = 0,1,2,...,n. \tag{5.1.11}$$

The orthogonality expression of $Q_{\Delta,m}^{(\tau_1,\tau_2)}(x,y)$ is as following

$$\int_0^{\Delta} \int_0^{\Delta} (Q_{\Delta,b}^{(\tau_1,\tau_2)}(x))(Q_{\Delta,a}^{(\tau_1,\tau_2)}(y))(Q_{\Delta,d}^{(\tau_1,\tau_2)}(x))(Q_{\Delta,c}^{(\tau_1,\tau_2)}(y)) R_{\Delta}^{(\tau_1,\tau_2)}(x) R_{\Delta}^{(\tau_1,\tau_2)}(y) dx dy = W_{\Delta,d}^{(\tau_1,\tau_2)} \delta_{a,d} W_{\Delta,c}^{(\tau_1,\tau_2)} \delta_{b,c}. \tag{5.1.12}$$

So, the function $r(x,y)$ defined and square integrable in the region $[0, \Delta] \times [0, \Delta]$ can be expressed in terms of the series of two–dimensional shifted JPs as follows

$$r(x,y) \approx \sum_{b=0}^{n} \sum_{a=0}^{n} D_{ba} (Q_{\Delta,b}^{(\tau_1,\tau_2)}(x))(Q_{\Delta,a}^{(\tau_1,\tau_2)}(y)), \tag{5.1.13}$$

D_{ba} is computed using the following relation

$$D_{ba} = \frac{1}{W_{\Delta,b}^{(\tau_1,\tau_2)} W_{\Delta,a}^{(\tau_1,\tau_2)}} \int_0^\Delta \int_0^\Delta r(x,y)(Q_{\Delta,b}^{(\tau_1,\tau_2)}(x))(Q_{\Delta,a}^{(\tau_1,\tau_2)}(y))R_\Delta^{(\tau_1,\tau_2)}(x,y)dxdy. \qquad (5.1.14)$$

The weight function $R_\Delta^{(\tau_1,\tau_2)}(x,y)$ is given by

$$R_\Delta^{(\tau_1,\tau_2)}(x,y) = R_\Delta^{(\tau_1,\tau_2)}(x)R_\Delta^{(\tau_1,\tau_2)}(y). \qquad (5.1.15)$$

(5.1.13) can also be expressed in the form of vector notation as

$$r(x,y) \approx \sum_{m=1}^{N^2} D_m Q_{\Delta,m}^{(\tau_1,\tau_2)}(x,y) = \hat{G}_{N^2}^T \hat{\Phi}_{N^2}(x,y), \quad \text{with } m = Nb+a+1, \text{ and } D_m = D_{ba}, \qquad (5.1.16)$$

where $\hat{G}_{N^2}$ refers to coefficient column vector of $N^2 \times 1$ dimensions and $\hat{\Phi}_{N^2}(x,y)$ indicates the column vector of dimensions $N^2 \times 1$, defined as

$$\hat{\Phi}_{N^2}(x,y) = \begin{pmatrix} \hat{\phi}_{11}(x,y) & \cdots & \hat{\phi}_{1N}(x,y) & \hat{\phi}_{21}(x,y) & \cdots & \hat{\phi}_{2N}(x,y) & \cdots & \hat{\phi}_{NN}(x,y) \end{pmatrix}^T,$$
$$(5.1.17)$$

where $\hat{\phi}_{j+1,i+1}(x,y) = (Q_{\Delta,j}^{(\tau_1,\tau_2)}(x))(Q_{\Delta,i}^{(\tau_1,\tau_2)}(y)), \; j,i = 0,1,2,...,n.$

5.1.2 Error Analysis

In this section we develop an analytical relationship for the error of approximation for sufficiently smooth function $r(x,y) \in C([0,\Delta] \times [0,\Delta])$. For this assume that $\prod_{N,N}(x,y)$ the space of N terms JPs and $g_{(N,N)}(x,y)$ is its best approximation in $\prod_{(N,N)}(x,y)$. Then according to the definition of the best approximation for any polynomial $f_{(N,N)}(x,y)$ of degree $\leq N$ in variable x and y, we can write

$$\|r(x,y) - r_{(N,N)}(x,y)\|_2 \leq \|r(x,y) - f_{(N,N)}(x,y)\|_2. \qquad (5.1.18)$$

Then by following the same procedure as adopted in [82], we can find the bounds for the errors given below

$$|r(x,y) - f_{(N,N)}(x,y)| \leq \frac{1}{4}\left(\frac{(\Delta-0)}{N}\right)^{N+1} max_{(x,y)}|\frac{\partial^{N+1}r(x,y)}{\partial x^{N+1}}|$$
$$+ \frac{1}{4}\left(\frac{(\Delta-0)}{N}\right)^{N+1} max_{(x,y)}|\frac{\partial^{N+1}r(x,y)}{\partial y^{N+1}}| \qquad (5.1.19)$$
$$+ \frac{1}{16}\left(\frac{(\Delta-0)}{N}\right)^{N+1}\left(\frac{(\Delta-0)}{N}\right)^{N+1} \max_{(x,y)}|\frac{\partial^{2N+2}}{\partial x^{N+1}\partial y^{N+1}}r(x,y)|.$$

Let $\hat{\lambda}_1 = max_{(x,y)}|\frac{\partial^{N+1} r(x,y)}{\partial x^{N+1}}|$, $\hat{\lambda}_2 = max_{(x,y)}|\frac{\partial^{N+1} r(x,y)}{\partial y^{N+1}}|$, and $\hat{\lambda}_3 = \max_{(x,y)}|\frac{\partial^{2N+2}}{\partial x^{N+1} \partial y^{N+1}} r(x,y)|$. Then, we have the following relation

$$
\begin{aligned}
|r(x,y) - f_{(N,N)}(x,y)| \leq &\frac{1}{4}\left(\frac{(\Delta - 0)}{N}\right)^{N+1} \hat{\lambda}_1 \\
&+ \frac{1}{4}\left(\frac{(\Delta - 0)}{N}\right)^{N+1} \hat{\lambda}_2 + \frac{1}{16}\left(\frac{(\Delta - 0)}{N}\right)^{N+1}\left(\frac{(\Delta - 0)}{N}\right)^{N+1} \hat{\lambda}_3.
\end{aligned}
\tag{5.1.20}
$$

Finally using (5.1.18), we have the following relation

$$
\begin{aligned}
\|r(x,y) - r_{(N,N)}(x,y)\|_2 \leq \sqrt[2]{(\Delta - 0)(\Delta - 0)}(&\frac{1}{4}\left(\frac{(\Delta - 0)}{N}\right)^{N+1} \hat{\lambda}_1 \\
&+ \frac{1}{4}\left(\frac{(\Delta - 0)}{N}\right)^{N+1} \hat{\lambda}_2 + \frac{1}{16}\left(\frac{(\Delta - 0)}{N}\right)^{N+1}\left(\frac{(\Delta - 0)}{N}\right)^{N+1} \hat{\lambda}_3).
\end{aligned}
\tag{5.1.21}
$$

For the detail study of the above analytical findings, we refer the reader to see [141].

The results of the next section are very important to develop the numerical scheme. Our main purpose in this section is the development of the new operational matrices of integrals and derivatives using two parametric orthogonal JPs. Using these operational matrices the fractional order problems (5.1.1)-(5.1.4) are reduced into a problems of solving a system of algebraic equations of Sylvester type which are simple in handling and can be solved by any computational software.

5.1.3 Operational matrices of fractional order integrals and derivatives

Lemma 5.1.1. *Let $\hat{\Phi}_{N^2}(x,y)$ be the function vector as defined in (5.1.17), then the fractional integration of order ρ of $\hat{\Phi}_{N^2}(x,y)$ with respect to x is given by*

$$
I_x^\rho(\hat{\Phi}_{N^2}(x,y)) \simeq \hat{Q}_{N^2 \times N^2}^{\rho,x} \hat{\Phi}_{N^2}(x,y),
\tag{5.1.22}
$$

where $\hat{Q}_{N^2 \times N^2}^{\rho,x}$ indicates the operational matrix of integration of order ρ and is defined as

$$
\hat{Q}_{N^2 \times N^2}^{\rho,x} = \begin{pmatrix}
\beth_{1,1,l} & \beth_{1,2,l} & \cdots & \beth_{1,q,l} & \cdots & \beth_{1,N^2,l} \\
\beth_{2,1,l} & \beth_{2,2,l} & \cdots & \beth_{2,q,l} & \cdots & \beth_{2,N^2,l} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\beth_{r,1,l} & \beth_{r,2,l} & \cdots & \beth_{r,q,l} & \cdots & \beth_{r,N^2,l} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\beth_{N^2,1,l} & \beth_{N^2,2,l} & \cdots & \beth_{N^2,q,l} & \cdots & \beth_{N^2,N^2,l}
\end{pmatrix},
$$

and $q = Nj + i + 1$, $r = Nb + 1 + 1$, $\beth_{r,q,l} = \daleth_{j,i,b,a,l}$ for $i,j,a,b = 0,1,2,...,m$,

$$
\daleth_{j,i,b,a,l} = \sum_{l=0}^{b} \biguplus_{b,l,\rho} d_{j,i,a},
$$

$$d_{jia} = \delta_{i,a} \sum_{k=0}^{j} \frac{(-1)^{j-k}(2j + \tau_1 + \tau_2 + 1)\Gamma(j+1)\Gamma(j + k + \tau_1 + \tau_2 + 1)\Gamma(l + \rho + k + \tau_2 + 1)\Gamma(\tau_1 + 1)\Delta^\rho}{\Gamma(j + \tau_1 + 1)\Gamma(k + \tau_2 + 1)(j-k)!k!\Gamma(l + \rho + k + \tau_2 + \tau_1 + 2)}.$$

And

$$\underset{b,l,\rho}{\boxed{+}} = \frac{(-1)^{b-l}\Gamma(b + \tau_2 + 1)\Gamma(b + l + \tau_1 + \tau_2 + 1)\Gamma(1+l)}{\Gamma(l + \tau_2 + 1)\Gamma(b + \tau_1 + \tau_2 + 1)(b-l)!l!\Gamma(1 + l + \rho)\Delta^l}. \tag{5.1.23}$$

Proof. To prove the above result take the fractional integration of (5.1.11) of order ρ with respect to x, we have

$$I_x^\rho Q_{\Delta,m}^{(\tau_1,\tau_2)}(x,y) = \sum_{l=0}^{b} \frac{(-1)^{b-l}\Gamma(b + \tau_2 + 1)\Gamma(b + l + \tau_1 + \tau_2 + 1)}{\Gamma(l + \tau_2 + 1)\Gamma(b + \tau_1 + \tau_2 + 1)(b-l)!l!\Delta^l} I_x^\rho x^l Q_{\Delta,a}^{(\tau_1,\tau_2)}(y).$$

Applying fractional integral, above relation becomes

$$I_x^\rho Q_{\Delta,m}^{(\tau_1,\tau_2)}(x,y) = \sum_{l=0}^{b} \frac{(-1)^{b-l}\Gamma(b + \tau_2 + 1)\Gamma(b + l + \tau_1 + \tau_2 + 1)\Gamma(1+l)}{\Gamma(l + \tau_2 + 1)\Gamma(b + \tau_1 + \tau_2 + 1)(b-l)!l!\Gamma(1 + l + \rho)\Delta^l} x^{l+\rho} P_{\Delta,a}^{(\tau_1,\tau_2)}(y). \tag{5.1.24}$$

The expression $x^{k+v} Q_{\Delta,a}^{(\tau_1,\tau_2)}(y)$ in view of two–dimensional JPs becomes

$$x^{l+\rho} Q_{\Delta,a}^{(\tau_1,\tau_2)}(y) \approx \sum_{j=0}^{n} \sum_{i=0}^{n} d_{j,i,a} Q_{\Delta,j}^{(\tau_1,\tau_2)}(x) Q_{\Delta,i}^{(\tau_1,\tau_2)}(y), \tag{5.1.25}$$

where $d_{j,i,a} = \frac{1}{W_{\Delta,j}^{(\tau_1,\tau_2)}, W_{\Delta,i}^{(\tau_1,\tau_2)}} \int_0^\Delta \int_0^\Delta x^{l+\rho} Q_{\Delta,a}^{(\tau_1,\tau_2)}(y) Q_{\Delta,j}^{(\tau_1,\tau_2)}(x) Q_{\Delta,i}^{(\tau_1,\tau_2)}(y) R_\Delta^{(\tau_1,\tau_2)}(x,y) dx dy$. And in the light of orthogonality condition, it may be written as

$$d_{j,i,a} = \frac{\delta_{i,a}}{W_{\Delta,j}^{(\tau_1,\tau_2)}} \int_0^\Delta x^{l+\rho} Q_{\Delta,j}^{(\tau_1,\tau_2)}(x) R_\Delta^{(\tau_1,\tau_2)}(x) dx. \tag{5.1.26}$$

With the help of (5.1.5) and (5.1.7), (5.1.26) can also be written as

$$d_{j,i,a} = \frac{\delta_{i,a}}{W_{\Delta,j}^{(\tau_1,\tau_2)}} \sum_{k=0}^{j} \frac{(-1)^{j-k}\Gamma(j + \tau_2 + 1)\Gamma(j + k + \tau_1 + \tau_2 + 1)}{\Gamma(k + \tau_2 + 1)\Gamma(j + \tau_1 + \tau_2 + 1)(j-k)!k!\Delta^k} \int_0^\Delta x^{l+\rho+k+\tau_2}(\Delta - x)^{\tau_1} dx. \tag{5.1.27}$$

Applying convolution theorem of Laplace transformation to the integrand of the above relation and then taking inverse Laplace of the obtained relation, we can write the generalized form of the coefficient involved in (5.1.25) as

$$d_{jia} = \frac{\delta_{i,a}}{W_{\Delta,j}^{(\tau_1,\tau_2)}} \sum_{k=0}^{j} \frac{(-1)^{j-k}\Gamma(j + \tau_2 + 1)\Gamma(j + k + \tau_1 + \tau_2 + 1)\Gamma(l + \rho + k + \tau_2 + 1)\Gamma(\tau_1 + 1)\Delta^{(l+\rho+k+\tau_2+\tau_1+1)}}{\Gamma(k + \tau_2 + 1)\Gamma(j + \tau_1 + \tau_2 + 1)(j-k)!k!\Delta^k\Gamma(l + \rho + k + \tau_2 + \tau_1 + 2)}. \tag{5.1.28}$$

Using (5.1.8) in (5.1.28) the most generalized form of the coefficient is as following

$$d_{jia} = \delta_{i,a} \sum_{k=0}^{j} \frac{(-1)^{j-k}(2j + \tau_1 + \tau_2 + 1)\Gamma(j+1)\Gamma(j + k + \tau_1 + \tau_2 + 1)\Gamma(l + \rho + k + \tau_2 + 1)\Gamma(\tau_1 + 1)\Delta^\rho}{\Gamma(j + \tau_1 + 1)\Gamma(k + \tau_2 + 1)(j-k)!k!\Gamma(l + \rho + k + \tau_2 + \tau_1 + 2)}. \tag{5.1.29}$$

Now, for convenience in handling the notations, let

$$\boxplus_{b,l,\rho} = \frac{(-1)^{b-l}\Gamma(b+\tau_2+1)\Gamma(b+l+\tau_1+\tau_2+1)\Gamma(1+l)}{\Gamma(l+\tau_2+1)\Gamma(b+\tau_1+\tau_2+1)(b-l)!l!\Gamma(1+l+\rho)\Delta^l}. \tag{5.1.30}$$

In the light of (5.1.25), (5.1.29) and (5.1.57), (5.1.24) can be written as

$$I_x^\rho Q_{\Delta,m}^{(\tau_1,\tau_2)}(x,y) = \sum_{l=0}^{b}\boxplus_{b,l,\rho}\sum_{j=0}^{n}\sum_{i=0}^{n} d_{j,i,a} Q_{\Delta,j}^{(\tau_1,\tau_2)}(x)Q_{\Delta,i}^{(\tau_1,\tau_2)}(y). \tag{5.1.31}$$

Let

$$\daleth_{j,i,b,a,l} = \sum_{l=0}^{b}\boxplus_{b,l,\rho} d_{j,i,a}, \tag{5.1.32}$$

we get

$$I_x^\rho Q_{\Delta,m}^{(\tau_1,\tau_2)}(x,y) = \sum_{j=0}^{n}\sum_{i=0}^{n}\daleth_{j,i,b,a,l} Q_{\Delta,j}^{(\tau_1,\tau_2)}(x)Q_{\Delta,i}^{(\tau_1,\tau_2)}(y). \tag{5.1.33}$$

Using the notations $q = Nj+i+1$, $r = Nb+a+1$, $\beth_{r,q,l} = \daleth_{j,i,a,b,l}$ for $i,j,a,b = 0,1,2,3,...,n$, we get the desired result. $\square$

Lemma 5.1.2. *Let $\hat{\Phi}_{N^2}(x,y)$ be the function vector as defined in (5.1.17), then the derivative of order τ of $\hat{\Phi}_{N^2}(x,y)$ with respect to x is as follows*

$$D_x^\tau(\hat{\Phi}_{N^2}(x,y)) \simeq \hat{D}_{N^2\times N^2}^{\tau,x}\hat{\Phi}_{N^2}(x,y), \tag{5.1.34}$$

where $\hat{D}_{N^2\times N^2}^{\tau,x}$ indicates the operational matrix of derivative of order τ and is defined as

$$\hat{D}_{N^2\times N^2}^{\tau,x} = \begin{pmatrix}
\beth_{1,1,l} & \beth_{1,2,l} & \cdots & \beth_{1,q,l} & \cdots & \beth_{1,N^2,l} \\
\beth_{2,1,l} & \beth_{2,2,l} & \cdots & \beth_{2,q,l} & \cdots & \beth_{2,N^2,l} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\beth_{r,1,l} & \beth_{r,2,l} & \cdots & \beth_{r,q,l} & \cdots & \beth_{r,N^2,l} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\beth_{N^2,1,l} & \beth_{N^2,2,l} & \cdots & \beth_{N^2,q,l} & \cdots & \beth_{N^2,N^2,l}
\end{pmatrix}, \tag{5.1.35}$$

and $r = Nj+i+1$, $q = Nb+c+1$, $\beth_{r,q,l} = \daleth_{j,i,b,a,l}$ for $i,j,a,b = 0,1,2,...,n$,

$$\daleth_{j,i,b,a,l} = \sum_{l=\lceil\tau\rceil}^{b}\boxplus_{b,l,\tau} d_{j,i,a}, \tag{5.1.36}$$

$$d_{jia} = \delta_{i,a}\sum_{k=0}^{j}\frac{(-1)^{j-k}(2j+\tau_1+\tau_2+1)\Gamma(j+1)\Gamma(j+k+\tau_1+\tau_2+1)\Gamma(l-\tau+k+\tau_2+1)\Gamma(\tau_1+1)\Delta^\tau}{\Gamma(j+\tau_1+1)\Gamma(k+\tau_2+1)(j-k)!k!\Gamma(l-\tau+k+\tau_1+\tau_2+2)}. \tag{5.1.37}$$

And

$$\boxplus_{b,l,\tau} = \frac{(-1)^{b-l}\Gamma(b+\tau_2+1)\Gamma(b+l+\tau_1+\tau_2+1)\Gamma(1+l)}{\Gamma(l+\tau_2+1)\Gamma(b+\tau_1+\tau_2+1)(b-l)!l!\Gamma(1+l-\tau)\Delta^l}. \tag{5.1.38}$$

Proof. To prove the above result take the fractional derivative of (5.1.11) of order τ with respect to x, we have

$$D_x^\tau Q_{\Delta,m}^{(\tau_1,\tau_2)}(x,y) == \sum_{l=0}^{b} \frac{(-1)^{b-l}\Gamma(b+\tau_2+1)\Gamma(b+l+\tau_1+\tau_2+1)}{\Gamma(l+\tau_2+1)\Gamma(b+\tau_1+\tau_2+1)(b-l)!l!\Delta^l} D_x^\tau x^l Q_{\Delta,a}^{(\tau_1,\tau_2)}(y),$$

In the light of the definition of the fractional derivative, above relation takes the following form

$$D_x^\tau Q_{\Delta,m}^{(\tau_1,\tau_2)}(x,y) = \sum_{l=\lceil\tau\rceil}^{b} \frac{(-1)^{b-l}\Gamma(b+\tau_2+1)\Gamma(b+l+\tau_1+\tau_2+1)\Gamma(1+l)}{\Gamma(l+\tau_2+1)\Gamma(b+\tau_1+\tau_2+1)(b-l)!l!\Gamma(1+l-\tau)\Delta^l} x^{l-\tau} Q_{\Delta,a}^{(\tau_1,\tau_2)}(y).$$

$$(5.1.39)$$

The expression $x^{l-\tau}Q_{\Delta,a}^{(\tau_1,\tau_2)}(y)$ in view of two–dimensional JPs becomes

$$x^{l-\tau}Q_{\Delta,a}^{(\tau_1,\tau_2)}(y) \approx \sum_{j=0}^{n}\sum_{i=0}^{n} d_{j,i,a} Q_{\Delta,j}^{(\tau_1,\tau_2)}(x)Q_{\Delta,i}^{(\tau_1,\tau_2)}(y), \qquad (5.1.40)$$

where $d_{j,i,a} = \frac{1}{W_{\Delta,j}^{(\tau_1,\tau_2)},W_{\Delta,i}^{(\tau_1,\tau_2)}} \int_0^\Delta \int_0^\Delta x^{l-\tau} Q_{\Delta,a}^{(\tau_1,\tau_2)}(y)Q_{\Delta,j}^{(\tau_1,\tau_2)}(x)Q_{\Delta,i}^{(\tau_1,\tau_2)}(y)R_\Delta^{(\tau_1,\tau_2)}(x,y)dxdy.$

And in the light of orthogonality condition, it may be written as

$$d_{j,i,a} = \frac{\delta_{i,a}}{W_{\Delta,j}^{(\tau_1,\tau_2)}} \int_0^\Delta x^{l-\tau}Q_{\Delta,j}^{(\tau_1,\tau_2)}(x)R_\Delta^{(\tau_1,\tau_2)}(x)dx. \qquad (5.1.41)$$

With the help of (5.1.5) and (5.1.7), (5.1.41) can be written as

$$d_{j,i,a} = \frac{\delta_{i,a}}{W_{\Delta,j}^{(\tau_1,\tau_2)}} \sum_{k=0}^{j} \frac{(-1)^{j-k}\Gamma(j+\tau_2+1)\Gamma(j+k+\tau_1+\tau_2+1)}{\Gamma(k+\tau_2+1)\Gamma(j+\tau_1+\tau_2+1)(j-k)!k!\Delta^k} \int_0^\Delta x^{l-\tau+k+\tau_2}(\Delta-x)^{\tau_1}. \quad (5.1.42)$$

Applying convolution theorem of Laplace transformation to the integrand of the above relation and then taking inverse Laplace of the obtained relation, we can write the generalized form of the coefficient involved in (5.1.40) as

$$d_{jia} = \delta_{i,a}\sum_{k=0}^{j} \frac{(-1)^{j-k}(2j+\tau_1+\tau_2+1)\Gamma(j+1)\Gamma(j+k+\tau_1+\tau_2+1)\Gamma(l-\tau+k+\tau_2+1)\Gamma(\tau_1+1)\Delta^{-\tau}}{\Gamma(j+\tau_1+1)\Gamma(k+\tau_2+1)(j-k)!k!\Gamma(l-\tau+k+\tau_2+\tau_1+2)}.$$

$$(5.1.43)$$

Now, for convenience in handling the notations, let

$$\biguplus_{b,l,\tau} = \frac{(-1)^{b-l}\Gamma(b+\tau_2+1)\Gamma(b+l+\tau_1+\tau_2+1)\Gamma(1+l)}{\Gamma(l+\tau_2+1)\Gamma(b+\tau_1+\tau_2+1)(b-l)!l!\Gamma(1+l-\tau)\Delta^l}. \qquad (5.1.44)$$

In the light of (5.1.40), (5.1.43),(5.1.44), (5.1.39) takes the form

$$D_x^\tau Q_{\Delta,m}^{(\tau_1,\tau_2)}(x,y) = \sum_{l=\lceil\tau\rceil}^{b} \biguplus_{b,l,\tau} \sum_{j=0}^{n}\sum_{i=0}^{n} d_{j,i,a} Q_{\Delta,i}^{(\tau_1,\tau_2)}(x)Q_{\Delta,i}^{(\tau_1,\tau_2)}(y). \qquad (5.1.45)$$

Let

$$\daleth_{j,i,b,a,l} = \sum_{l=\lceil\tau\rceil}^{b} \biguplus_{b,l,\tau} E_{j,i,a}, \qquad (5.1.46)$$

we get

$$D_x^\tau Q_{\Delta,m}^{(\tau_1,\tau_2)}(x,y) = \sum_{j=0}^{n}\sum_{i=0}^{n} \daleth_{j,i,b,a,l} Q_{\Delta,i}^{(\tau_1,\tau_2)}(x) Q_{\Delta,i}^{(\tau_1,\tau_2)}(y). \qquad (5.1.47)$$

Using the notations $q = Nj+i+1$, $r = Nb+a+1$, $\daleth_{r,q,l} = \daleth_{j,i,a,b,l}$ for i, j, a, $b = 0,1,2,3,...,n$, we get the desired result. $\qquad \square$

Lemma 5.1.3. *let* $\hat{\Phi}_{N^2}(x,y)$ *be the function vector as defined in* (5.1.17), *then the derivative of order* τ *of* $\hat{\Phi}_{N^2}(x,y)$ *with respect to* y *is as follows*

$$D_y^\tau(\hat{\Phi}_{N^2}(x,y)) \simeq \hat{D}_{N^2\times N^2}^{\tau,x} \hat{\Phi}_{N^2}(x,y), \qquad (5.1.48)$$

where $\hat{D}_{N^2\times N^2}^{\tau,x}$ *indicates the operational matrix of derivative of order* τ *and is defined as*

$$\hat{D}_{N^2\times N^2}^{\tau,y} = \begin{pmatrix} \daleth_{1,1,l} & \daleth_{1,2,l} & \cdots & \daleth_{1,q,l} & \cdots & \daleth_{1,N^2,l} \\ \daleth_{2,1,l} & \daleth_{2,2,l} & \cdots & \daleth_{2,q,l} & \cdots & \daleth_{2,N^2,l} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \daleth_{r,1,l} & \daleth_{r,2,l} & \cdots & \daleth_{r,q,l} & \cdots & \daleth_{r,N^2,l} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \daleth_{N^2,1,l} & \daleth_{N^2,2,l} & \cdots & \daleth_{N^2,q,l} & \cdots & \daleth_{N^2,N^2,l} \end{pmatrix}, \qquad (5.1.49)$$

and $r = Nj+i+1$, $q = Nb+c+1$, $\daleth_{r,q,l} = \varpi_{j,i,b,c,l}$ *for* $j,i,b,c = 0,1,2,...,n$,

$$\varpi_{j,i,b,c,l} = \sum_{l=\lceil\tau\rceil}^{b} \Lambda_{b,l,\tau}\chi_{j,i,c}, \qquad (5.1.50)$$

$$\chi_{jic} = \delta_{j,b}\sum_{m=0}^{j} \frac{(-1)^{j-m}(2j+\tau_1+\tau_2+1)\Gamma(j+1)\Gamma(j+m+\tau_1+\tau_2+1)\Gamma(l-\tau+m+\tau_2+1)\Gamma(\tau_1+1)\Delta^\tau}{\Gamma(j+\tau_1+1)\Gamma(m+\tau_2+1)(j-m)!m!\Gamma(l-\tau+l+\tau_2+\tau_1+2)}, \qquad (5.1.51)$$

and

$$\Lambda_{b,l,\tau} = \frac{(-1)^{b-l}\Gamma(b+\tau_2+1)\Gamma(b+l+\tau_1+\tau_2+1)\Gamma(1+l)}{\Gamma(l+\tau_2+1)\Gamma(b+\tau_1+\tau_2+1)(b-l)!l!\Gamma(1+l-\tau)\Delta^k}. \qquad (5.1.52)$$

Proof. The proof of this lemma is similar as Lemma 5.1.2. $\qquad \square$

Lemma 5.1.4. *Let* $\hat{\Phi}_{N^2}(x,y)$ *be the function vector as defined in* (5.1.17), *then the mixed partial derivative of order* τ *of* $\hat{\Phi}_{N^2}(x,y)$ *is as follows*

$$\frac{\partial^\tau}{\partial x^{(\tau/2)}\partial y^{(\tau/2)}}(\hat{\Phi}_{N^2}(x,y)) \simeq \hat{D}_{N^2\times N^2}^{\tau,x,y} \hat{\Phi}_{N^2}(x,y), \qquad (5.1.53)$$

where $\hat{D}_{N^2\times N^2}^{\tau,x,y}$ *is the operational matrix of mixed partial derivatives of order* τ, *and is defined as*

$$\hat{D}_{N^2\times N^2}^{\hat{}x,y} = \begin{pmatrix} \daleth_{1,1,l,m} & \daleth_{1,2,l,m} & \cdots & \daleth_{1,q,l,m} & \cdots & \daleth_{1,N^2,l,m} \\ \daleth_{2,1,l,m} & \daleth_{2,2,l,m} & \cdots & \daleth_{2,q,l,m} & \cdots & \daleth_{2,N^2,l,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \daleth_{r,1,l,m} & \daleth_{r,2,l,m} & \cdots & \daleth_{r,q,l,m} & \cdots & \daleth_{r,N^2,l,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \daleth_{N^2,1,l,m} & \daleth_{N^2,2,l,m} & \cdots & \daleth_{N^2,q,l,m} & \cdots & \daleth_{N^2,N^2,l,m} \end{pmatrix}, \qquad (5.1.54)$$

and $q = Nj + i + 1$, $r = Nb + c + 1$, $\beth_{r,q,l,m} = \varpi_{j,i,b,c,l,m,T}$ *for $j, i, b, c = 0, 1, 2, ..., n$,*

$$\varpi_{j,i,b,c,l,m,\Delta} = \sum_{l=\lceil\frac{\tau}{2}\rceil}^{b} \sum_{m=\lceil\frac{\tau}{2}\rceil}^{c} \widehat{\Theta}_{(b,l,\tau_1,\tau_2,\Delta)} \widehat{\Theta'}_{(c,m,\tau_1,\tau_2,\Delta)} \chi_{(j,i,l,m)}, \tag{5.1.55}$$

$$\chi_{(j,i,l,m)} = \frac{1}{W_{T,j}^{(\tau_1,\tau_2)} W_{\Delta,i}^{(\tau_1,\tau_2)}} \sum_{m'=0}^{j} \sum_{l'=0}^{i} \Theta_{(j,m',\tau_1,\tau_2,\Delta)} \Theta'_{(i,l',\tau_1,\tau_2,\Delta)} \Omega_{(m',l,\tau_1,\tau_2,\Delta)} \Omega'_{(l',m,\tau_1,\tau_2,\Delta)}, \tag{5.1.56}$$

$$\Theta_{(j,m',\tau_1,\tau_2,\Delta)} = \frac{(-1)^{j-m'}\Gamma(j+\tau_2+1)\Gamma(j+m'+\tau_1+\tau_2+1)}{\Gamma(m'+\tau_2+1)\Gamma(j+\tau_1+\tau_2+1)(b-m')!m'!\Delta^{m'}}, \tag{5.1.57}$$

$$\Omega_{(m',l,\tau_1,\tau_2,\Delta)} = \frac{\Gamma(m'+\tau_2+l-\tau/2+1)\Gamma(\tau_1+1)}{\Gamma(m'+\tau_2+l-\tau/2+\tau_1+1)} \Delta^{m'+\tau_2+l-\tau/2+\tau_1+1}, \tag{5.1.58}$$

and

$$\widehat{\Theta}_{(b,l,\tau_1,\tau_2,\Delta)} = \frac{(-1)^{b-l}\Gamma(b+\tau_2+1)\Gamma(b+l+\tau_1+\tau_2+1)}{\Gamma(l+\tau_2+1)\Gamma(b+\tau_1+\tau_2+1)(b-l)!\Delta^l\Gamma(1+k-\tau/2)}. \tag{5.1.59}$$

Proof. To prove the result take the fractional derivative of order τ of $Q_{\Delta,n}^{(\tau_1,\tau_2)}(x,y)$ as defined in (5.1.11), the following relation is obtained

$$\frac{\partial^\tau}{\partial x^{(\tau/2)}\partial y^{(\tau/2)}} Q_{\Delta,n}^{(\tau_1,\tau_2)}(x,y) = \sum_{l=0}^{b} \Theta_{(b,l,\tau_1,\tau_2,T)} \sum_{m=0}^{c} \Theta'_{(c,m,\tau_1,\tau_2,T)} D_x^{\tau/2} x^l D_y^{\tau/2} y^m,$$

where

$$\Theta_{(b,l,\tau_1,\tau_2,T)} = \frac{(-1)^{b-l}\Gamma(b+\tau_2+1)\Gamma(b+l+\tau_1+\tau_2+1)}{\Gamma(l+\tau_2+1)\Gamma(b+\tau_1+\tau_2+1)(b-l)!l!\Delta^l},$$

$$\frac{\partial^\tau}{\partial x^{(\tau/2)}\partial y^{(\tau/2)}} Q_{\Delta,n}^{(\tau_1,\tau_2)}(x,y) = \sum_{l=\lceil\frac{\tau}{2}\rceil}^{b} \Theta_{(b,l,\tau_1,\tau_2,\Delta)} \sum_{m=\lceil\frac{\tau}{2}\rceil}^{c} \Theta'_{(c,m,\tau_1,\tau_2,\Delta)} \times$$
$$\frac{\Gamma(1+l)\Gamma(1+m)}{\Gamma(1+l-\tau/2)\Gamma(1+m-\tau/2)} x^{(l-\tau/2)} y^{(m-\tau/2)}, \tag{5.1.60}$$

The series approximation of $x^{(l-\tau/2)}y^{(m-\tau/2)}$ with two dimensional JPs, yields

$$x^{(l-\tau/2)}y^{(m-\tau/2)} = \sum_{j=0}^{n} \sum_{i=0}^{n} \chi_{(j,i,l,m)} Q_{\Delta,j}^{(\tau_1,\tau_2)}(x) Q_{\Delta,i}^{(\tau_1,\tau_2)}(y), \tag{5.1.61}$$

where

$$\chi_{(j,i,l,m)} = \frac{1}{W_{\Delta,j}^{(\tau_1,\tau_2)} W_{\Delta,i}^{(\tau_1,\tau_2)}} \int_0^\Delta \int_0^\Delta x^{(l-\tau/2)} y^{(m-\tau/2)} Q_{\Delta,j}^{(\tau_1,\tau_2)}(x) Q_{\Delta,i}^{(\tau_1,\tau_2)}(y) R_\Delta^{(\tau_1,\tau_2)}(x,y) dx dy. \tag{5.1.62}$$

Above relation further implies that

$$\chi_{(j,i,l,m)} = \frac{1}{W_{\Delta,j}^{(\tau_1,\tau_2)} W_{\Delta,i}^{(\tau_1,\tau_2)}} \sum_{m'=0}^{j} \sum_{l'=0}^{i} \Theta_{(j,m',\tau_1,\tau_2,\Delta)} \Theta'_{(i,l',\tau_1,\tau_2,\Delta)} \int_0^\Delta y^{(l'+\tau_2+m-\tau/2)} (\Delta-y)_1^\tau dy$$
$$\int_0^\Delta x^{(m'+\tau_2+m-\tau/2)} (\Delta-x)_1^\tau dx. \tag{5.1.63}$$

Applying convolution theorem of Laplace transformation to the integrands of the above relation and then taking inverse Laplace of the obtained relations, we have

$$\int_0^\Delta x^{(m'+\tau_2+l-\tau/2)}(\Delta-x)^{\tau_1}dx = \frac{\Gamma(m'+\tau_2+l-\tau/2+1)\Gamma(\tau_1+1)\Delta^{(m'+\tau_2+l-\tau/2+\tau_1+1)}}{\Gamma(m'+\tau_2+l-\tau/2+\tau_1+1)},$$

and

$$\int_0^\Delta y^{(l'+\tau_2+m-\tau/2)}(\Delta-y)^{\tau_1}dy = \frac{\Gamma(l'+\tau_2+m-\tau/2+1)\Gamma(\tau_1+1)\Delta^{(l'+\tau_2+m-\tau/2+\tau_1+1)}}{\Gamma(l'+\tau_2+m-\tau/2+\tau_1+1)}.$$

With the help of above relations, the coefficient $\chi_{(j,i,l,m)}$ of (5.1.61) can be written in a more compact form as

$$\chi_{(j,i,l,m)} = \frac{1}{W_{\Delta,j}^{(\tau_1,\tau_2)}W_{\Delta,i}^{(\tau_1,\tau_2)}}\sum_{m'=0}^{j}\sum_{l'=0}^{i}\Theta_{(j,m',\tau_1,\tau_2,\Delta)}\Theta'_{(i,l',\tau_1,\tau_2,\Delta)}\Omega_{(m',l,\tau_1,\tau_2,\Delta)}\Omega'_{(l',m,\tau_1,\tau_2,\Delta)},$$

$$(5.1.64)$$

where

$$\Omega_{(m',l,\tau_1,\tau_2,\Delta)} = \frac{\Gamma(m'+\tau_2+l-\tau/2+1)\Gamma(\tau_1+1)}{\Gamma(m'+\tau_2+l-\tau/2+\tau_1+1)}\Delta^{m'+\tau_2+l-\tau/2+\tau_1+1}, \qquad (5.1.65)$$

and

$$\Omega'_{(l',m,\tau_1,\tau_2,\Delta)} = \frac{\Gamma(l'+\tau_2+m-\tau/2+1)\Gamma(\tau_1+1)}{\Gamma(l'+\tau_2+m-\tau/2+\tau_1+1)}\Delta^{l'+\tau_2+m-\tau/2+\tau_1+1}. \qquad (5.1.66)$$

Now using (5.1.64) along with (5.1.61),(5.1.65) and (5.1.66) in equation (5.1.60), we get

$$\frac{\partial^\tau Q_{\Delta,b}^{(\tau_1,\tau_2)}(x)Q_{\Delta,c}^{(\tau_1,\tau_2)}(y)}{\partial x^{(\tau/2)}\partial y^{(\tau/2)}} = \sum_{j=0}^{n}\sum_{i=0}^{n}\sum_{l=\lceil\frac{\tau}{2}\rceil}^{b}\sum_{m=\lceil\frac{\tau}{2}\rceil}^{c}\widehat{\Theta}_{(b,l,\tau_1,\tau_2,\Delta)}\widehat{\Theta'}_{(c,m,\tau_1,\tau_2,\Delta)}\chi_{(j,i,l,m)}Q_{\Delta,j}^{(\tau_1,\tau_2)}(x)Q_{\Delta,i}^{(\tau_1,\tau_2)}(y),$$

$$(5.1.67)$$

where

$$\widehat{\Theta}_{(b,l,\tau_1,\tau_2,\Delta)} = \frac{(-1)^{b-l}\Gamma(b+\tau_2+1)\Gamma(b+l+\tau_1+\tau_2+1)}{\Gamma(l+\tau_2+1)\Gamma(b+\tau_1+\tau_2+1)(b-l)!\Delta^l\Gamma(1+l-\tau/2)}, \qquad (5.1.68)$$

and

$$\widehat{\Theta'}_{(a,m,\tau_1,\tau_2,\Delta)} = \frac{(-1)^{a-m}\Gamma(a+\tau_2+1)\Gamma(a+m+\tau_1+\tau_2+1)}{\Gamma(m+\tau_2+1)\Gamma(a+\tau_1+\tau_2+1)(a-m)!\Delta^m\Gamma(1+m-\tau/2)}. \qquad (5.1.69)$$

Let

$$\varpi_{j,i,b,c,l,m,\Delta} = \sum_{l=\lceil\frac{\tau}{2}\rceil}^{b}\sum_{m=\lceil\frac{\tau}{2}\rceil}^{c}\widehat{\Theta}_{(b,l,\tau_1,\tau_2,\Delta)}\widehat{\Theta'}_{(a,m,\tau_1,\tau_2,\Delta)}\chi_{(j,i,l,m)}, \qquad (5.1.70)$$

we get

$$\frac{\partial^\tau}{\partial x^{(\tau/2)}\partial y^{(\tau/2)}}Q_{\Delta,b}^{(\tau_1,\tau_2)}(x)Q_{\Delta,c}^{(\tau_1,\tau_2)}(y) = \sum_{j=0}^{n}\sum_{i=0}^{n}\varpi_{j,i,b,a,l,m,\Delta}Q_{\Delta,j}^{(\tau_1,\tau_2)}(x)Q_{\Delta,i}^{(\tau_1,\tau_2)}(y). \qquad (5.1.71)$$

Using the notations $q = Nj+i+1$, $r = Nb+c+1$, $\beth_{r,q,l,m} = \varpi_{j,i,c,b,l,m,T}$ for i, j, b, $c = 0, 1, 2, 3, ..., n$, we get the desired result. $\qquad\square$

5.2 Applications of the operational matrices of integration and derivative of FOPDEs

In this section we are interested to generalize a new numerical technique to approximate the solutions of a generalized class of multi terms FOPDEs and its coupled systems.

5.2.1 Solution of multi-terms of FOPDEs by JPs

Firstly, we develop the numerical technique for the multi-terms FOPDEs with mixed partial derivatives of fractional order of the form

$$\frac{\partial^{\sigma_1} U(x,y)}{\partial x^{\sigma_1}} = a_1 \frac{\partial^{\gamma_1} U(x,y)}{\partial y^{\gamma_1}} + a_2 \frac{\partial^{\varrho_1} U(x,y)}{\partial x^{\varrho_1/2} \partial y^{\varrho_1/2}} + G_1(x,y), \tag{5.2.1}$$

corresponding to the following initial conditions

$$U^{(j)}(0,y) = H_j(y), \ j = 0,1...,m. \tag{5.2.2}$$

We are interested in finding the solution of (5.2.1) in the form of orthogonal two-parametric shifted JPs, such that the following exists

$$\frac{\partial^{\sigma_1} U(x,y)}{\partial x^{\sigma_1}} = \mathbf{K}_{N^2} \hat{\Phi}_{N^2}(x,y). \tag{5.2.3}$$

By application of fractional integral of order σ and making use of operational matrix of integration, above expression can be written as

$$U(x,y) - \sum_{j=0}^{m} c_j x^j = \mathbf{K}_{N^2} \hat{\mathbf{Q}}_{N^2 \times N^2}^{(\sigma,x)} \hat{\Phi}_{N^2}(x,y). \tag{5.2.4}$$

Using (5.2.2), we have

$$U(x,y) = \mathbf{K}_{N^2} \hat{\mathbf{Q}}_{N^2 \times N^2}^{(\sigma,x)} \hat{\Phi}_{N^2}(x,y) + \mathbf{F}_{N^2}^1 \hat{\Phi}_{N^2}(x,y). \tag{5.2.5}$$

By application of operational matrices of derivatives, we have

$$\frac{\partial^{\gamma_1} U(x,y)}{\partial y^{\gamma_1}} = \mathbf{K}_{N^2} \hat{\mathbf{Q}}_{N^2 \times N^2}^{(\sigma,x)} \hat{\mathbf{D}}_{N^2 \times N^2}^{(\gamma_1,y)} \hat{\Phi}_{N^2}(x,y) + \mathbf{F}_{N^2}^1 \hat{\mathbf{D}}_{N^2 \times N^2}^{(\gamma_1,y)} \hat{\Phi}_{N^2}(x,y), \tag{5.2.6}$$

$$\frac{\partial^{\varrho_1} U(x,y)}{\partial x^{\varrho_1/2} \partial y^{\varrho_1/2}} = \mathbf{K}_{N^2} \hat{\mathbf{Q}}_{N^2 \times N^2}^{(\sigma,x)} \hat{\mathbf{D}}_{N^2 \times N^2}^{(\varrho_1,x,y)} \hat{\Phi}_{N^2}(x,y) + \mathbf{F}_{N^2}^1 \hat{\mathbf{D}}_{N^2 \times N^2}^{(\varrho_1,x,y)} \hat{\Phi}_{N^2}(x,y). \tag{5.2.7}$$

Now using the estimates (5.2.7),(5.2.6) and (5.2.4) in (5.2.1) we get the following system of algebraic equations

$$\mathbf{K}_{N^2} = \mathbf{K}_{N^2}\hat{\mathbf{Q}}^{(\sigma,x)}_{N^2\times N^2}(a_1\hat{\mathbf{D}}^{(\gamma_1,y)}_{N^2\times N^2} + a_2\hat{\mathbf{D}}^{(\varrho_1,x,y)}_{N^2\times N^2}) + \mathbf{F}^1_{N^2}(a_1\hat{\mathbf{D}}^{(\gamma_1,y)}_{N^2\times N^2} + a_2\hat{\mathbf{D}}^{(\varrho_1,x,y)}_{N^2\times N^2}) + \mathbf{F}^2_{N^2}. \quad (5.2.8)$$

The unknown matrix $\mathbf{K}_{N^2}$ can be easily computed by solving the above system of algebraic equations.

5.2.2 Solution of Coupled systems of multi-terms FOPDEs by JPs

Now we develop the numerical technique for the following coupled system of multi-terms FOPDEs with mixed partial derivative terms of fractional order as

$$\begin{aligned}
\frac{\partial^{\sigma_1}U(x,y)}{\partial x^{\sigma_1}} =&a_1\frac{\partial^{\gamma_1}U(x,y)}{\partial y^{\gamma_1}} + a_2\frac{\partial^{\gamma_2}V(x,y)}{\partial x^{\gamma_2}} + a_3\frac{\partial^{\gamma_3}V(x,y)}{\partial y^{\gamma_3}} +\\
&a_4\frac{\partial^{\varrho_4}U(x,y)}{\partial x^{\varrho_4/2}\partial y^{\varrho_4/2}} + a_5\frac{\partial^{\varrho_3}V(x,y)}{\partial x^{\varrho_3/4}\partial y^{\varrho_3/2}} + G_1(x,y),\\
\frac{\partial^{\sigma_2}V(x,y)}{\partial x^{\sigma_2}} =&b_1\frac{\partial^{\rho_1}V(x,y)}{\partial y^{\rho_1}} + b_2\frac{\partial^{\rho_2}U(x,y)}{\partial x^{\rho_2}} + b_3\frac{\partial^{\rho_3}U(x,y)}{\partial y^{\rho_3}} +\\
&b_4\frac{\partial^{\varrho_1}U(x,y)}{\partial x^{\varrho_1/2}\partial y^{\varrho_1/2}} + b_5\frac{\partial^{\varrho_2}V(x,y)}{\partial x^{\varrho_2/2}\partial y^{\varrho_2/2}} + G_2(x,y),
\end{aligned} \quad (5.2.9)$$

corresponding to the following initial conditions

$$U^{(j)}(0,y) = h_j(y),\ V^{(j)}(0,y) = g_j(y),\ j = 0,1...,m, \quad (5.2.10)$$

where $m < \sigma_1, \sigma_2 \leq m+1$, $a_j, b_j,\ j = 1,2,...,5$ are all real constants and $U = U(x,y)$, $V = V(x,y)$ are the unknown solutions of the system to be determined. Moreover $U(x,y), V(x,y) \in C([0,\Delta] \times [0,\Delta])$, and $F_1(x,y), F_2(x,y) \in C([0,\Delta] \times [0,\Delta])$. We are interested in finding the solution of the problem (5.2.9) in the form of orthogonal shifted JPs, such that the following exists

$$\frac{\partial^{\sigma_1}U(x,y)}{\partial x^{\sigma_1}} = \mathbf{K}_{N^2}\hat{\Phi}_{N^2}(x,y) \quad ,\frac{\partial^{\sigma_2}V(x,y)}{\partial x^{\sigma_2}} = \mathbf{L}_{N^2}\hat{\Phi}_{N^2}(x,y). \quad (5.2.11)$$

Now applying integral of order σ_1 and σ_2 on the corresponding equations, we get

$$\begin{aligned}
U(x,y) - \sum_{j=0}^{m}c_jx^j &= \mathbf{K}_{N^2}\hat{\mathbf{Q}}^{(\sigma_1,x)}_{N^2\times N^2}\hat{\Phi}_{N^2}(x,y),\\
V(x,y) - \sum_{j=0}^{m}d_jx^j &= \mathbf{L}_{N^2}\hat{\mathbf{Q}}^{(\sigma_2,x)}_{N^2\times N^2}\hat{\Phi}_{N^2}(x,y).
\end{aligned} \quad (5.2.12)$$

The generalized relations for the solution of the problem (5.2.9) and constants of integration can be obtained by applying the initial conditions (5.2.10) as

$$U(x,y) = \mathbf{K}_{N^2}\hat{\mathbf{Q}}^{(\sigma_1,x)}_{N^2 \times N^2}\hat{\Phi}_{N^2}(x,y) + \mathbf{F}^1_{N^2}\hat{\Phi}_{N^2}(x,y),$$
$$V(x,y) = \mathbf{L}_{N^2}\hat{\mathbf{Q}}^{(\sigma_2,x)}_{N^2 \times N^2}\hat{\Phi}_{N^2}(x,y) + \mathbf{F}^2_{N^2}\hat{\Phi}_{N^2}(x,y),$$

(5.2.13)

where $\mathbf{F}^1_{N^2}\hat{\Phi}_{N^2}(x,y) = \sum_{j=0}^{m} h_j(y)x^j$, $\mathbf{F}^2_{N^2}\hat{\Phi}_{N^2}(x,y) = \sum_{j=0}^{m} g_j(y)x^j$. For simplicity of notations we can write

$$\mathbf{K}_{N^2}\hat{\mathbf{Q}}^{(\sigma_1,x)}_{N^2 \times N^2} + \mathbf{F}^1_{N^2} = \widetilde{\mathbf{K}_{N^2}}, \qquad \mathbf{L}_{N^2}\hat{\mathbf{Q}}^{(\sigma_1,x)}_{N^2 \times N^2} + \mathbf{F}^1_{N^2} = \widetilde{\mathbf{L}_{N^2}}.$$

(5.2.14)

Further, it can also be written as

$$U(x,y) = \widetilde{\mathbf{K}_{N^2}}\hat{\Phi}_{N^2}(x,y), \qquad V(x,y) = \widetilde{\mathbf{L}_{N^2}}\hat{\Phi}_{N^2}(x,y).$$

(5.2.15)

Using relation (5.2.15) and operational matrices of derivatives we can get the following estimates

$$\frac{\partial^{\gamma_1}U(x,y)}{\partial y^{\gamma_1}} = \widetilde{\mathbf{K}_{N^2}}\hat{\mathbf{D}}^{(\gamma_1,y)}_{N^2 \times N^2}\hat{\Phi}_{N^2}(x,y), \quad \frac{\partial^{\gamma_2}V(x,y)}{\partial x^{\gamma_2}} = \widetilde{\mathbf{L}_{N^2}}\hat{\mathbf{D}}^{(\gamma_2,x)}_{N^2 \times N^2}\hat{\Phi}_{N^2}(x,y),$$

$$\frac{\partial^{\gamma_3}V(x,y)}{\partial y^{\gamma_3}} = \widetilde{\mathbf{L}_{N^2}}\hat{\mathbf{D}}^{(\gamma_3,y)}_{N^2 \times N^2}\hat{\Phi}_{N^2}(x,y) \quad, \frac{\partial^{\varrho_4}U(x,y)}{\partial x^{\varrho_4/2}\partial y^{\varrho_4/2}} = \widetilde{\mathbf{K}_{N^2}}\hat{\mathbf{D}}^{(\varrho_4,x,y)}_{N^2 \times N^2}\hat{\Phi}_{N^2}(x,y),$$

$$\frac{\partial^{\varrho_3}V(x,y)}{\partial x^{\varrho_3/2}\partial y^{\varrho_3/2}} = \widetilde{\mathbf{L}_{N^2}}\hat{\mathbf{D}}^{(\varrho_3,x,y)}_{N^2 \times N^2}\hat{\Phi}_{N^2}(x,y), \quad \frac{\partial^{\rho_1}V(x,y)}{\partial y^{\rho_1}} = \widetilde{\mathbf{L}_{N^2}}\hat{\mathbf{D}}^{(\rho_1,y)}_{N^2 \times N^2}\hat{\Phi}_{N^2}(x,y),$$

(5.2.16)

$$\frac{\partial^{\rho_2}U(x,y)}{\partial x^{\rho_2}} = \widetilde{\mathbf{K}_{N^2}}\hat{\mathbf{D}}^{(\rho_2,x)}_{N^2 \times N^2}\hat{\Phi}_{N^2}(x,y), \quad \frac{\partial^{\rho_3}U(x,y)}{\partial y^{\rho_3}} = \widetilde{\mathbf{K}_{N^2}}\hat{\mathbf{D}}^{(\rho_3,y)}_{N^2 \times N^2}\hat{\Phi}_{N^2}(x,y),$$

$$\frac{\partial^{\varrho_1}U(x,y)}{\partial x^{\varrho_1/2}\partial y^{\varrho_1/2}} = \widetilde{\mathbf{K}_{N^2}}\hat{\mathbf{D}}^{(\varrho_1,x,y)}_{N^2 \times N^2}\hat{\Phi}_{N^2}(x,y), \quad \frac{\partial^{\varrho_2}V(x,y)}{\partial x^{\varrho_2/2}\partial y^{\varrho_2/2}} = \widetilde{\mathbf{L}_{N^2}}\hat{\mathbf{D}}^{(\varrho_2,x,y)}_{N^2 \times N^2}\hat{\Phi}_{N^2}(x,y),$$

$$F_1(x,y) = \mathbf{F}^3_{N^2}\hat{\Phi}_{N^2}(x,y), \qquad F_2(x,y) = \mathbf{F}^4_{N^2}\hat{\Phi}_{N^2}(x,y).$$

$$\begin{pmatrix} \mathbf{K}_{N^2}\hat{\Phi}_{N^2}(x,y) \\ \mathbf{L}_{N^2}\hat{\Phi}_{N^2}(x,y) \end{pmatrix} = \begin{pmatrix} a_1\widetilde{\mathbf{K}_{N^2}}\hat{\mathbf{D}}^{(\gamma_1,y)}_{N^2 \times N^2}\hat{\Phi}_{N^2}(x,y) \\ b_1\widetilde{\mathbf{L}_{N^2}}\hat{\mathbf{D}}^{(\rho_1,y)}_{N^2 \times N^2}\hat{\Phi}_{N^2}(x,y) \end{pmatrix} + \begin{pmatrix} a_2\widetilde{\mathbf{L}_{N^2}}\hat{\mathbf{D}}^{(\gamma_2,x)}_{N^2 \times N^2}\hat{\Phi}_{N^2}(x,y) \\ b_2\widetilde{\mathbf{K}_{N^2}}\hat{\mathbf{D}}^{(\rho_2,x)}_{N^2 \times N^2}\hat{\Phi}_{N^2}(x,y) \end{pmatrix}$$

$$+ \begin{pmatrix} a_3\widetilde{\mathbf{L}_{N^2}}\hat{\mathbf{D}}^{(\gamma_3,y)}_{N^2 \times N^2}\hat{\Phi}_{N^2}(x,y) \\ b_3\widetilde{\mathbf{K}_{N^2}}\hat{\mathbf{D}}^{(\rho_3,y)}_{N^2 \times N^2}\hat{\Phi}_{N^2}(x,y) \end{pmatrix} + \begin{pmatrix} a_4\widetilde{\mathbf{K}_{N^2}}\hat{\mathbf{D}}^{(\varrho_4,x,y)}_{N^2 \times N^2}\hat{\Phi}_{N^2}(x,y) \\ b_5\widetilde{\mathbf{L}_{N^2}}\hat{\mathbf{D}}^{(\varrho_2,x,y)}_{N^2 \times N^2}\hat{\Phi}_{N^2}(x,y) \end{pmatrix} + \begin{pmatrix} \mathbf{F}^3_{N^2}\hat{\Phi}_{N^2}(x,y) \\ \mathbf{F}^4_{N^2}\hat{\Phi}_{N^2}(x,y) \end{pmatrix}$$

$$+ \begin{pmatrix} a_5\widetilde{\mathbf{L}_{N^2}}\hat{\mathbf{D}}^{(\varrho_3,x,y)}_{N^2 \times N^2}\hat{\Phi}_{N^2}(x,y) \\ b_4\widetilde{\mathbf{K}_{N^2}}\hat{\mathbf{D}}^{(\varrho_1,x,y)}_{N^2 \times N^2}\hat{\Phi}_{N^2}(x,y) \end{pmatrix}.$$

(5.2.17)

$$\left(\begin{array}{cc} \mathbf{K}_{N^2} & \mathbf{L}_{N^2} \end{array}\right)\hat{A} = \left(\begin{array}{cc} \widetilde{\mathbf{K}_{N^2}} & \widetilde{\mathbf{L}_{N^2}} \end{array}\right)\left(\begin{array}{cc} a_1\hat{\mathbf{D}}_{N^2\times N^2}^{(\gamma_1,y)} & O_{N^2\times N^2} \\ O_{N^2\times N^2} & b_1\hat{\mathbf{D}}_{N^2\times N^2}^{(\rho_1,y)} \end{array}\right)\hat{A}$$

$$+ \left(\begin{array}{cc} \widetilde{\mathbf{K}_{N^2}} & \widetilde{\mathbf{L}_{N^2}} \end{array}\right)\left(\begin{array}{cc} O_{N^2\times N^2} & b_2\hat{\mathbf{D}}_{N^2\times N^2}^{(\rho_2,x)} \\ a_2\hat{\mathbf{D}}_{N^2\times N^2}^{(\gamma_2,x)} & O_{N^2\times N^2} \end{array}\right)\hat{A}$$

$$+ \left(\begin{array}{cc} \widetilde{\mathbf{K}_{N^2}} & \widetilde{\mathbf{L}_{N^2}} \end{array}\right)\left(\begin{array}{cc} O_{N^2\times N^2} & b_3\hat{\mathbf{D}}_{N^2\times N^2}^{(\rho_3,y)} \\ a_3\hat{\mathbf{D}}_{N^2\times N^2}^{(\gamma_3,y)} & O_{N^2\times N^2} \end{array}\right)\hat{A}$$

$$+ \left(\begin{array}{cc} \widetilde{\mathbf{K}_{N^2}} & \widetilde{\mathbf{L}_{N^2}} \end{array}\right)\left(\begin{array}{cc} a_4\hat{\mathbf{D}}_{N^2\times N^2}^{(\varrho_4,x,y)} & O_{N^2\times N^2} \\ O_{N^2\times N^2} & b_5\hat{\mathbf{D}}_{N^2\times N^2}^{(\varrho_2,x,y)} \end{array}\right)\hat{A}$$

$$\left(\begin{array}{cc} \widetilde{\mathbf{K}_{N^2}} & \widetilde{\mathbf{L}_{N^2}} \end{array}\right)\left(\begin{array}{cc} O_{N^2\times N^2} & b_4\hat{\mathbf{D}}_{N^2\times N^2}^{(\varrho_1,x,y)} \\ a_5\hat{\mathbf{D}}_{N^2\times N^2}^{(\varrho_3,x,y)} & O_{N^2\times N^2} \end{array}\right)\hat{A} + \left(\begin{array}{cc} \mathbf{F}_{N^2}^3 & \mathbf{F}_{N^2}^4 \end{array}\right)\hat{A},$$

$$(5.2.18)$$

where

$$\hat{A} = \left(\begin{array}{cc} \hat{\Phi}_{N^2}(x,y) & O_{N^2} \\ O_{N^2} & \hat{\Phi}_{N^2}(x,y) \end{array}\right).$$

O_{N^2} is a column vector of zeros, and $O_{N^2\times N^2}$ is matrix having all entries equal to zero.

Now canceling out the common term and after a short simplification we can write (5.2.18) as

$$\left(\begin{array}{cc} \mathbf{K}_{N^2} & \mathbf{L}_{N^2} \end{array}\right) - \left(\begin{array}{cc} \widetilde{\mathbf{K}_{N^2}} & \widetilde{\mathbf{L}_{N^2}} \end{array}\right)\widehat{H} - \left(\mathbf{F}_{N^2}^3\mathbf{F}_{N^2}^4\right) = 0, \qquad (5.2.19)$$

where

$$\widehat{H} = \left(\begin{array}{cc} a_1\hat{\mathbf{D}}_{N^2\times N^2}^{(\gamma_1,y)} + a_4\hat{\mathbf{D}}_{N^2\times N^2}^{(\varrho_4,x,y)} & b_2\hat{\mathbf{D}}_{N^2\times N^2}^{(\rho_2,x)} + b_3\hat{\mathbf{D}}_{N^2\times N^2}^{(\rho_3,y)} + b_4\hat{\mathbf{D}}_{N^2\times N^2}^{(\varrho_1,x,y)} \\ a_2\hat{\mathbf{D}}_{N^2\times N^2}^{(\gamma_2,x)} + a_3\hat{\mathbf{D}}_{N^2\times N^2}^{(\gamma_3,y)} + a_5\hat{\mathbf{D}}_{N^2\times N^2}^{(\varrho_3,x,y)} & b_5\hat{\mathbf{D}}_{N^2\times N^2}^{(\varrho_2,x,y)} + b_1\hat{\mathbf{D}}_{N^2\times N^2}^{(\rho_1,y)} \end{array}\right).$$

Using the values of $\widetilde{\mathbf{K}_{N^2}}$ and $\widetilde{\mathbf{L}_{N^2}}$ we can write the above equations as

$$\left(\begin{array}{cc} \mathbf{K}_{N^2} & \mathbf{L}_{N^2} \end{array}\right) - \left(\begin{array}{cc} \mathbf{K}_{N^2} & \mathbf{L}_{N^2} \end{array}\right)\widehat{H_1} - \left(\mathbf{F}_{N^2}^1\mathbf{F}_{N^2}^2\right)\widehat{H} - \left(\mathbf{F}_{N^2}^3\mathbf{F}_{N^2}^4\right) = 0, \qquad (5.2.20)$$

where

$$\widehat{H_1} = \left(\begin{array}{cc} \hat{\mathbf{Q}}_{N^2\times N^2}^{(\sigma_1,x)}\mathbf{D_1} & \hat{\mathbf{Q}}_{N^2\times N^2}^{(\sigma_1,x)}\mathbf{D_2} \\ \hat{\mathbf{Q}}_{N^2\times N^2}^{(\sigma_1,x)}\mathbf{D_3} & \hat{\mathbf{Q}}_{N^2\times N^2}^{(\sigma_1,x)}\mathbf{D_4} \end{array}\right), \mathbf{D_1} = (a_1\hat{\mathbf{D}}_{N^2\times N^2}^{(\gamma_1,y)} + a_4\hat{\mathbf{D}}_{N^2\times N^2}^{(\varrho_4,x,y)}),$$

$\mathbf{D_2} = (b_2\hat{\mathbf{D}}_{N^2\times N^2}^{(\rho_2,x)} + b_3\hat{\mathbf{D}}_{N^2\times N^2}^{(\rho_3,y)} + b_4\hat{\mathbf{D}}_{N^2\times N^2}^{(\varrho_1,x,y)})$, $\mathbf{D_3} = (a_2\hat{\mathbf{D}}_{N^2\times N^2}^{(\gamma_2,x)} + a_3\hat{\mathbf{D}}_{N^2\times N^2}^{(\gamma_3,y)} + a_5\hat{\mathbf{D}}_{N^2\times N^2}^{(\varrho_3,x,y)})$, and

$\mathbf{D_4} = (b_5\hat{\mathbf{D}}_{N^2\times N^2}^{(\varrho_2,x,y)} + b_1\hat{\mathbf{D}}_{N^2\times N^2}^{(\rho_1,y)})$.

The unknowns $(\mathbf{K}_{N^2}\ \mathbf{L}_{N^2})$ can be easily determined using (5.2.20) which is a system of easily solvable algebraic equations. The approximate solution of the problem (5.2.9)–(5.2.10) can be easily computed by inserting $(\mathbf{K}_{N^2}\ \mathbf{L}_{N^2})$ into (5.2.15).

5.2.3   Numerical test of the proposed scheme

By solving some test problems, the validity of the proposed method is examined in this section. Note that all the simulations are carried out using 5 Ghz processor. All the results are displayed using plots and tables. The algorithm is designed in such a way that it can be easily simulated using any computational software. We use $MATLAB$ for calculation and simulations.

Example 5.2.1. *As a first test case consider the following FOPDEs with mixed partial derivative terms*

$$\frac{\partial^{\sigma_1} U(x,y)}{\partial x^{\sigma_1}} = a_1 \frac{\partial^{\gamma_1} U(x,y)}{\partial y^{\gamma_1}} + a_2 \frac{\partial^{\varrho_1} U(x,y)}{\partial x^{\frac{\varrho_1}{2}} \partial y^{\frac{\varrho_1}{2}}} + G_1(x,y),$$ (5.2.21)

where source term G_1 is given as under

$G_1(x,y) = \frac{39239842694707435\, x^{\frac{1}{5}}\, \sin(y)}{18014398509481984} - x^2\, \cos(y) - 2\, x\, \cos(y).$

Example 5.2.2. *As a second test case assume the integer order coupled problem of PDEs of the form*

$$\begin{aligned}
\frac{\partial^2 U}{\partial x^2} &= 2\frac{\partial U}{\partial y} + 2\frac{\partial V}{\partial x} + 3\frac{\partial V}{\partial y} + 5\frac{\partial^2 U}{\partial x \partial y} + 6\frac{\partial^2 V}{\partial x \partial y} + G_1(x,y), \\
\frac{\partial^2 V}{\partial x^2} &= 3\frac{\partial V}{\partial y} + 4\frac{\partial U}{\partial x} + 2\frac{\partial U}{\partial y} + 3\frac{\partial^2 U}{\partial x \partial y} - \frac{\partial^2 V}{\partial x \partial y} + G_2(x,y),
\end{aligned}$$ (5.2.22)

where the source terms G_1 and G_2 are defined as

$G_1 = -8\, x^4\, y^3 - 80\, x^3\, y^3 + 15\, x^3\, y^2 + 12\, x^2\, y^4 + 6\, x^2\, y^3 + 99\, x^2\, y^2 - 6\, x^2\, y - 6\, x\, y^3 - 4\, x\, y^2 - 24\, x\, y,$ (5.2.23)

$G_2 = -8\, x^4\, y^3 - 16\, x^3\, y^4 - 48\, x^3\, y^3 + 15\, x^3\, y^2 + 12\, x^2\, y^3 + 18\, x^2\, y^2 - 6\, x^2\, y - 6\, x\, y^3 + 4\, x\, y + 2\, y^2.$ (5.2.24)

corresponding to the following initial conditions

$$U(0,y) = U'(0,y) = 0, \qquad V(0,y) = V'(0,y) = 0.$$ (5.2.25)

Example 5.2.3. *As a third test case consider the following coupled system of FOPDEs*

$$\begin{aligned}
\frac{\partial^{1.8} U}{\partial x^{1.8}} &= \frac{\partial^{0.8} U}{\partial y^{0.8}} + \frac{\partial^2 U}{\partial x \partial y} + \frac{\partial^2 V}{\partial x \partial y} + G_1(x,y), \\
\frac{\partial^{1.8} V}{\partial x^{1.8}} &= \frac{\partial^{0.8} V}{\partial y^{0.8}} + \frac{\partial^2 U}{\partial x \partial y} + \frac{\partial^2 V}{\partial x \partial y} + G_2(x,y),
\end{aligned}$$ (5.2.26)

where source terms G_1 and G_2 are given below

$$\begin{aligned}
G_1(x,y) =\, &3\, x^2\, y^3\, (2\, y - 2)\, (x - 1)^3 + 3\, x^3\, y^3\, (2\, y - 2)\, (x - 1)^2 + 4\, x^4\, y^3\, (2\, x - 2)\, (y - 1)^3 + \\
&3\, x^4\, y^4\, (2\, x - 2)\, (y - 1)^2 + 9\, x^2\, y^2\, (x - 1)^3\, (y - 1)^2 + 9\, x^3\, y^2\, (x - 1)^2\, (y - 1)^2 + \\
&16\, x^3\, y^3\, (x - 1)^2\, (y - 1)^3 + 12\, x^3\, y^4\, (x - 1)^2\, (y - 1)^2 - \\
&\frac{89181460669789625\, x^{\frac{11}{5}}\, y^4\, (y - 1)^3\, (125\, x^2 - 175\, x + 56)}{504403158265495552} + \\
&\frac{445907303348948125\, x^4\, y^{\frac{16}{5}}\, (x - 1)^2\, (4375\, y^3 - 11625\, y^2 + 10075\, y - 2821)}{406548945561989414912},
\end{aligned}$$ (5.2.27)

$$
\begin{aligned}
G_2(x,y) =& 3\,x^2\,y^3\,(2\,y-2)\,(x-1)^3 + 3\,x^3\,y^3\,(2\,y-2)\,(x-1)^2 + 12\,x^4\,y^3\,(2\,x-2)\,(y-1)^3 + \\
& 9\,x^4\,y^4\,(2\,x-2)\,(y-1)^2 + 9\,x^2\,y^2\,(x-1)^3\,(y-1)^2 + 9\,x^3\,y^2\,(x-1)^2\,(y-1)^2 + \\
& 36\,x^3\,y^4\,(x-1)^2\,(y-1)^2 + \frac{89181460669789625\,x^3\,y^{\frac{11}{5}}\,(x-1)^3\,\left(125\,y^2 - 210\,y + 84\right)}{3026418949592973312} \\
& -\frac{17836292133957925\,x^{\frac{6}{5}}\,y^3\,(y-1)^2\,\left(1250\,x^3 - 2625\,x^2 + 1680\,x - 308\right)}{1008806316530991104} \\
& + 48\,x^3\,y^3\,(x-1)^2\,(y-1)^3.
\end{aligned}
$$

$$(5.2.28)$$

Example 5.2.4. *As a fourth test case consider the following coupled system of FOPDEs*

$$
\begin{aligned}
\frac{\partial^{1.8} U}{\partial x^{1.8}} =& \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} + \frac{\partial^2 V}{\partial x \partial y} + G_1(x,y), \\
\frac{\partial^2 V}{\partial x^2} =& \frac{\partial^{0.8} U}{\partial x^{0.8}} + \frac{\partial U}{\partial y} + \frac{\partial^2 U}{\partial x \partial y} + G_2(x,y),
\end{aligned}
$$

$$(5.2.29)$$

where the source terms G_1 and G_2 are given as under

$$
G_1(x,y) = \frac{39239842694707435\,x^{\frac{1}{5}}\,\mathrm{e}^y}{18014398509481984} - y^2\,\mathrm{e}^x - x^2\,\mathrm{e}^y - 2\,y\,\mathrm{e}^x,
$$

$$(5.2.30)$$

$$
G_2(x,y) = y^2\,\mathrm{e}^x - x^2\,\mathrm{e}^y - 8\,x\,\mathrm{e}^y.
$$

$$(5.2.31)$$

5.2.4 Results and Discussions

The exact solution for the first test Example 5.2.1 for $\sigma_1 = 1.8$, $\gamma_1 = 1$, and $\varrho_1 = 2$ with $a_1 = 1$, and $a_2 = 2$ is $U(x,y) = x^2 sin(y)$. At first glance we compare the approximate solution of (5.2.1) with the exact solution of (5.2.1) by fixing $\sigma_1 = 1.8$, $\gamma_1 = 1$, and $\varrho_1 = 2$ with $\tau_1 = 2 = \tau_2$, the parameters of JPs at scale level $M = 5$. We observe that at scale level $M = 5$, our approximate solution are in a good agreement with the exact solution as displayed in Figure 5.1. We also investigate the error analysis at different scale levels and observe that the absolute errors between exact and approximate solution is minimizing as the scale levels are increasing. The results are displayed in Figure 5.2. We fix $\gamma_1 = 1$, $\varrho_1 = 2$ and observe the behaviour of the approximate solutions of (5.2.1) for different values of σ_1, it is found that the approximate solutions for different values of σ_1 approaching to the exact solution as $\sigma_1 \to 3$. The results are displayed in Figure 5.3.

The exact solutions of Example 5.2.2 are given as under

$$
U(x,y) = (xy)^4 - (xy)^3, \quad V(x,y) = (xy)^2 - (xy)^3.
$$

We approximate the solution of the second test Example 5.2.2 of integer order at different scale levels. We compare exact and approximate solution of the problem depicting in Figures (5.4) and

(5.5) for parameters $\tau_1 = 2$, $\tau_2 = 2$ and scale level $M = 5$, observing that the exact solution matches well with the approximate solution. We also approximate the absolute error at different scale level $M = 5, 6$, and 7 and at different points of the plane showing in Tables (5.1) and (5.2). Figure (5.6) showing that with the increase of the scale level the error of approximation (absolute error) decreases significantly. We also observe the behaviour of the numerical scheme for different values of σ_1 and find that as $\sigma_1 \to 2$, the approximate solutions approach to the exact solution. The results are displayed in Figures 5.7 and 5.8.

The exact solution of Example 5.2.3 is given below that can be easily verified by direct substitution

$$U(x,y) = -x^4\, y^4\, (x-1)^2\, (y-1)^3, \quad V(x,y) = -x^3\, y^3\, (x-1)^3\, (y-1)^2.$$

We analyze the test Example 5.2.3 using our proposed numerical scheme and observe that the numerical results are in a good agreement with the exact solution. At different scale levels we check the validity of our numerical results and examine that our proposed numerical scheme exhibits great resemblance with the exact solution. We carry out the error analysis and note that with the increase in scale level the absolute error decreases in a great extent. We display the exact and approximate solution of (5.2.3) with parameters $\tau_1 = 1$, $\tau_2 = 1$ and scale level $M = 6$. The results are displayed at Figures 5.9 and 5.10. The results regarding to error analysis with parameters $\tau_1 = 1$, $\tau_2 = 1$ and scale level $M = 10$ are displayed in Figures 5.11 and 5.12.

The exact solution of fractional order test Example 5.2.4 is given below that can be easily verified by direct substitution

$$U(x,y) = x^2\, e^y, \quad V(x,y) = y^2\, e^x.$$

We investigate the numerical accuracy of our proposed scheme with the help of Example 5.2.4 at different scale levels and examine that the proposed numerical scheme provides highly efficient and accurate results. We note that results generated by proposed scheme are highly accurate even for low scale level. That shows the accuracy of our developed scheme. We also do the error analysis and note that the absolute error decreases remarkably with the increase in scale level. The results obtained using the proposed scheme are compared with the exact solution of (5.2.4) choosing parameters $\tau_1 = 1$, and $\tau_2 = 2$ at scale level $M = 7$ and note that results are in a good agreement with the exact solutions as displayed in Figures 5.13 and 5.14. The graphical results regarding to the error
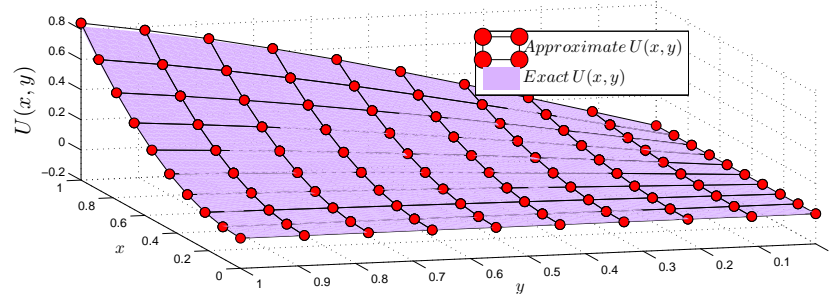
Figure 5.1: Comparing approximate solution with the exact solution of Example 5.2.1.

Table 5.1: At various scale level M and for different values of x and y, the absolute error for Example 5.2.2 is computed for the unknown function $U(x, y)$.

$(x, y) \setminus M$	$M = 5$	$M = 6$	$M = 7$
$(x, y) = (0.1, 0.1)$	$1.707E - 13$	$1.425E - 13$	$1.567E - 13$
$(x, y) = (0.1, 0.5)$	$1.956E - 13$	$1.596E - 13$	$1.235E - 13$
$(x, y) = (0.1, 0.9)$	$2.236E - 13$	$1.773E - 13$	$1.756E - 12$
$(x, y) = (0.5, 0.1)$	$1.873E - 12$	$6.380E - 13$	$1.857E - 13$
$(x, y) = (0.5, 0.5)$	$1.880E - 12$	$7.858E - 13$	$1.648E - 12$
$(x, y) = (0.5, 0.9)$	$2.092E - 13$	$9.619E - 13$	$1.829E - 12$
$(x, y) = (0.9, 0.1)$	$1.270E - 11$	$9.363E - 12$	$1.472E - 11$
$(x, y) = (0.9, 0.5)$	$1.307E - 11$	$1.029E - 11$	$1.497E - 13$
$(x, y) = (0.9, 0.9)$	$1.306E - 11$	$1.418E - 12$	$1.652E - 13$

analysis with parameters $\tau_1 = 1$, and $\tau_2 = 2$ at scale level $M = 10$ are displayed in Figures 5.15 and 5.16.

5.3   Multi-terms partial differential equations of fractional order in high dimensions

The main aim of this section is to extend the idea for high dimensional multi-terms FOPDEs. The new operational matrices are developed for a function vector involving three variables and with the help of these operational matrices, we solve the FOPDEs in high dimensions. A multi-term three dimensional heat conduction problem is chosen to check the efficiency and applicability of the proposed numerical scheme.
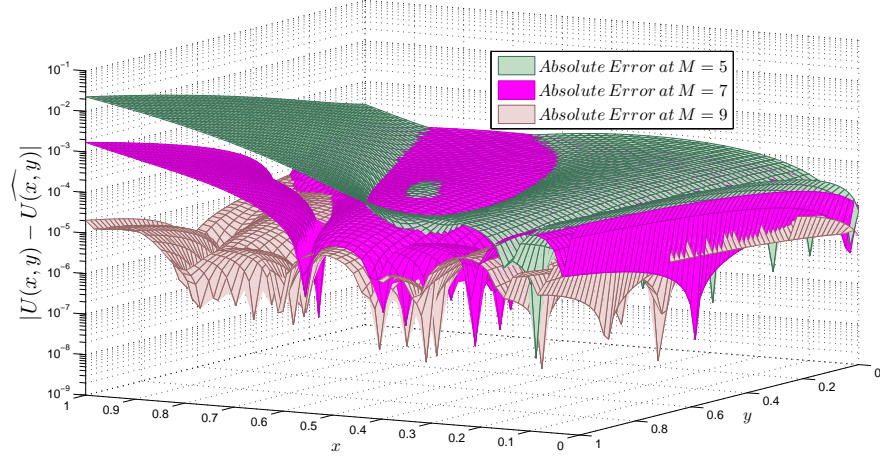
Figure 5.2: For various values of M (scale level), absolute error is computed for Example 5.2.1.
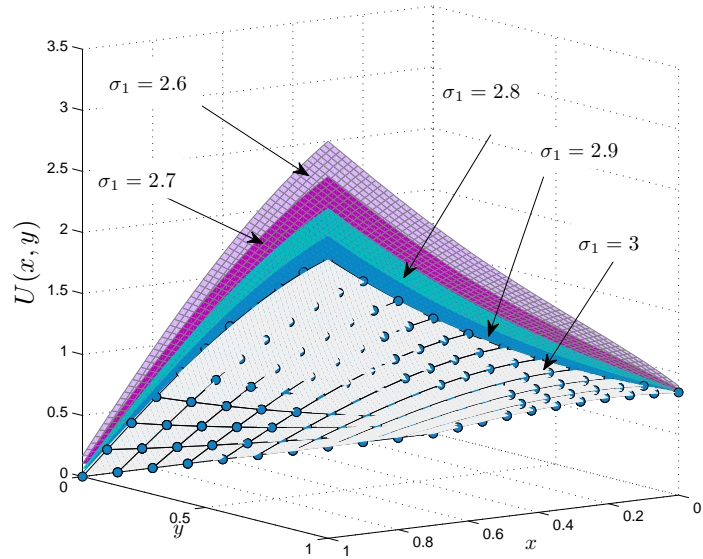

Figure 5.3: For multiple values of σ_1, the comparison is presented between approximate and exact solutions of Example 5.2.1.
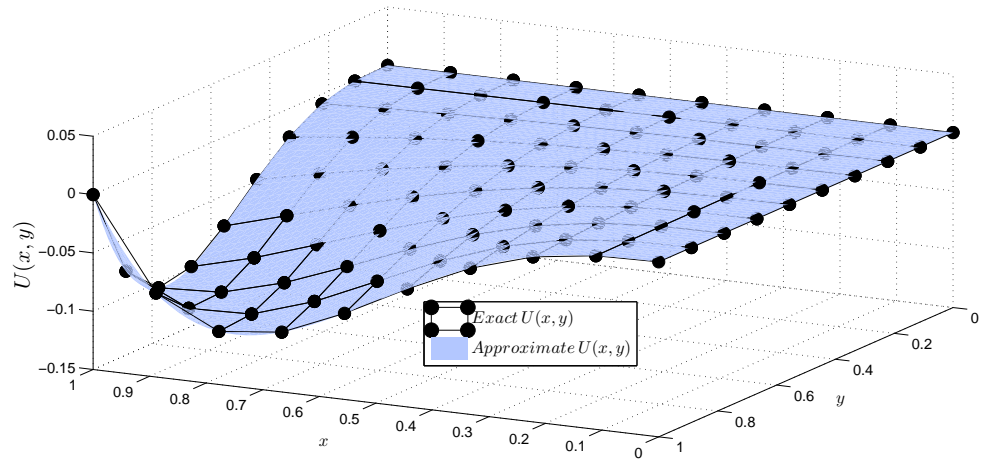
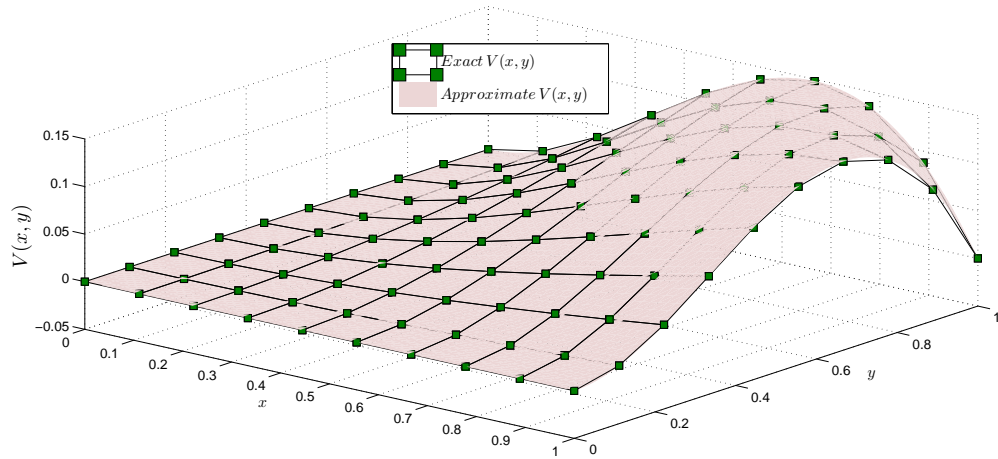Figure 5.4: Comparing approximate solution with the exact solution of Example 5.2.2.

Figure 5.5: The comparison between approximate and exact solutions of Example 5.2.2 is presented.
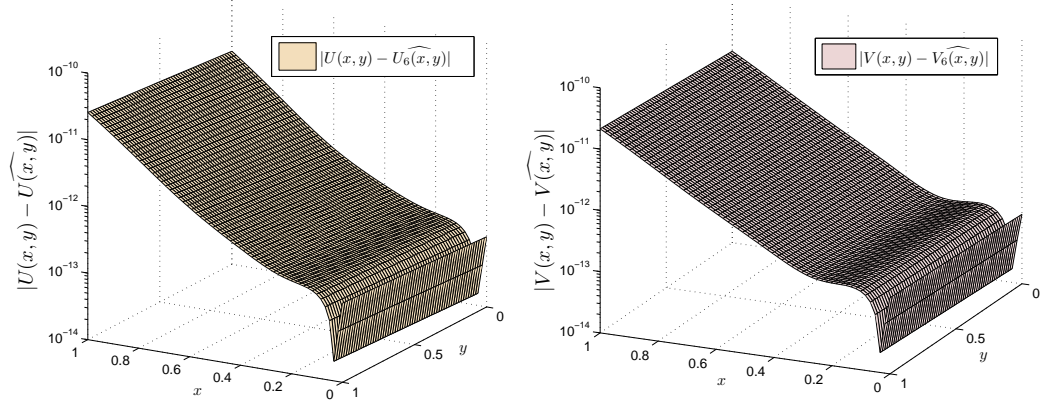
Figure 5.6: Error of approximation in $U(x,y)$ and $V(x,y)$ of Example 5.2.2

Table 5.2: At various scale level M and for different values of x and y, the absolute error for Example 5.2.2 is computed for the unknown function $V(x,y)$.

$(x,y) \setminus M$	$M = 5$	$M = 6$	$M = 7$
$(x,y) = (0.1, 0.1)$	$1.826E - 13$	$1.678E - 13$	$1.231E - 13$
$(x,y) = (0.1, 0.5)$	$2.638E - 13$	$1.839E - 13$	$1.578E - 13$
$(x,y) = (0.1, 0.9)$	$2.126E - 13$	$2.019E - 13$	$1.378E - 13$
$(x,y) = (0.5, 0.1)$	$4.451E - 12$	$6.368E - 13$	$1.582E - 12$
$(x,y) = (0.5, 0.5)$	$3.781E - 12$	$7.520E - 13$	$1.673E - 12$
$(x,y) = (0.5, 0.9)$	$3.933E - 12$	$9.051E - 13$	$1.567E - 11$
$(x,y) = (0.9, 0.1)$	$1.650E - 11$	$8.394E - 12$	$1.482E - 12$
$(x,y) = (0.9, 0.5)$	$1.820E - 11$	$9.996E - 12$	$1.785E - 13$
$(x,y) = (0.9, 0.9)$	$1.746E - 11$	$1.209E - 11$	$1.652E - 13$
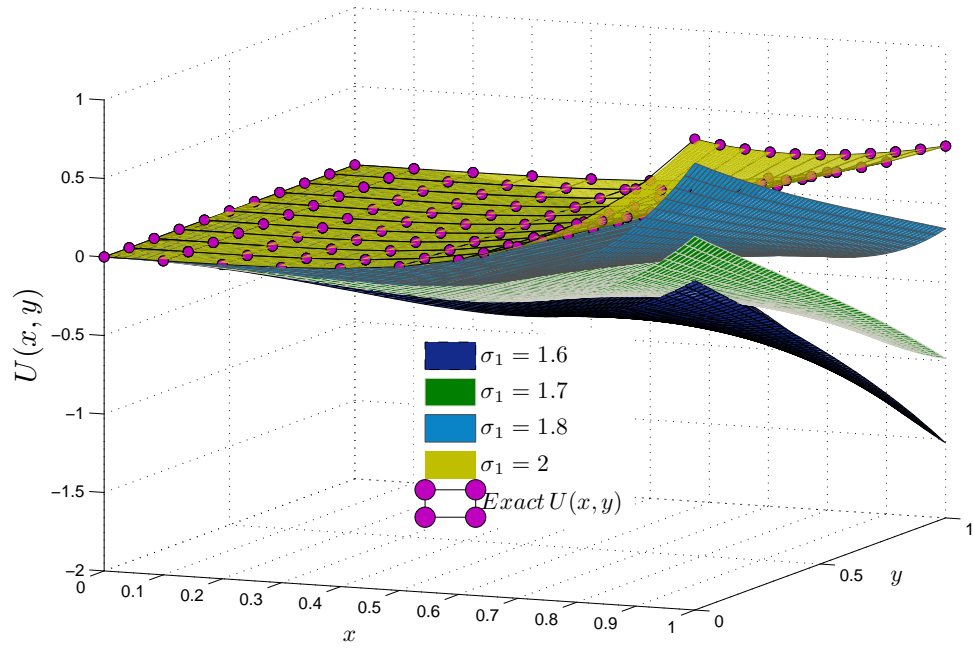
Figure 5.7: For multiple values of σ_1, the comparison between approximate and exact solutions of Example 5.2.2 is done.

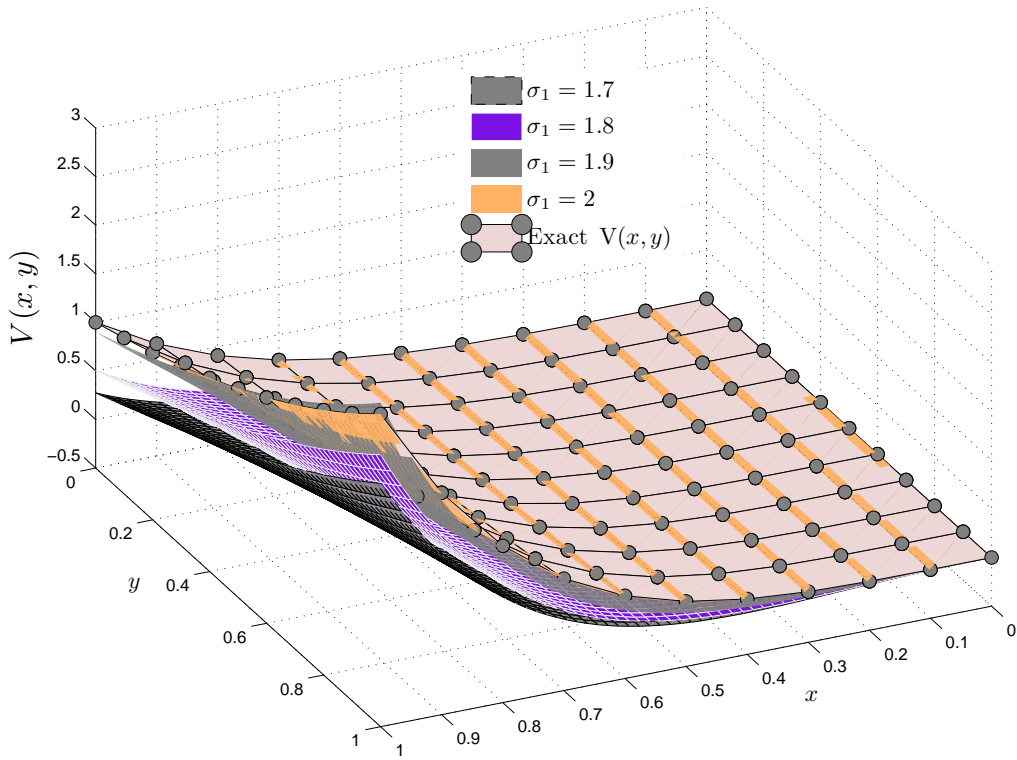Figure 5.8: For multiple values of σ_1, the comparison between approximate and exact solutions of Example 5.2.2 is done.
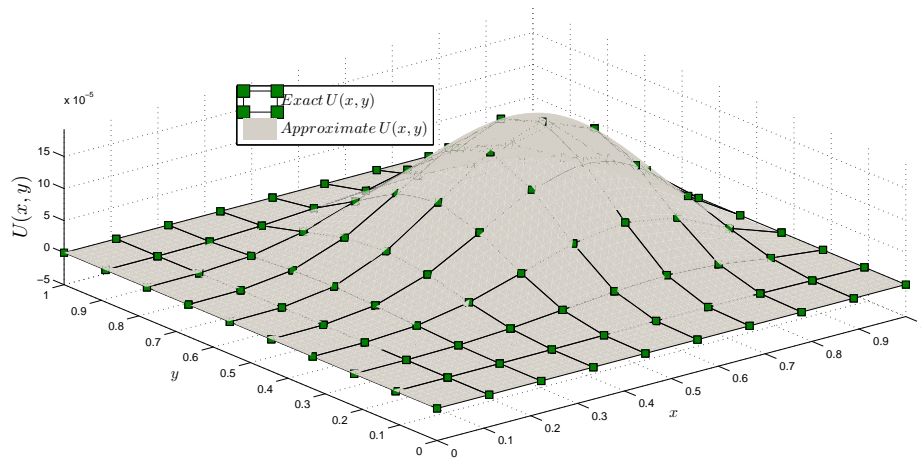

Figure 5.9: Comparison is given between approximate solution and exact solution of Example 5.2.3
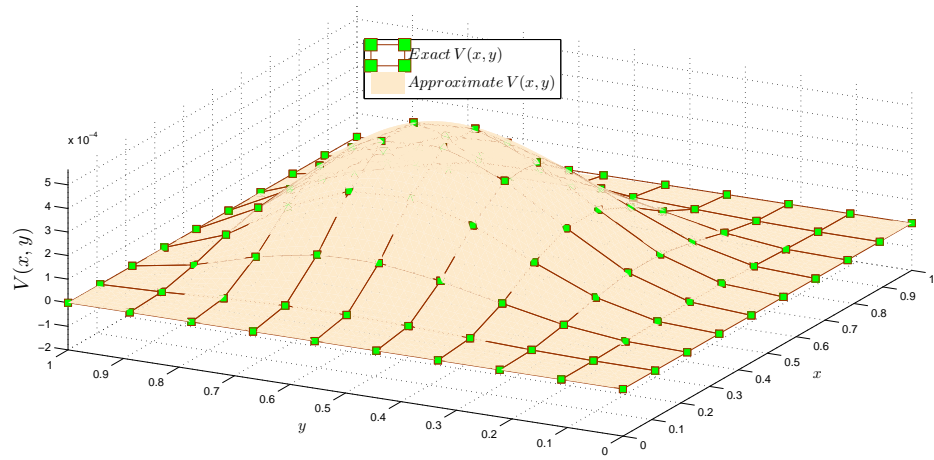
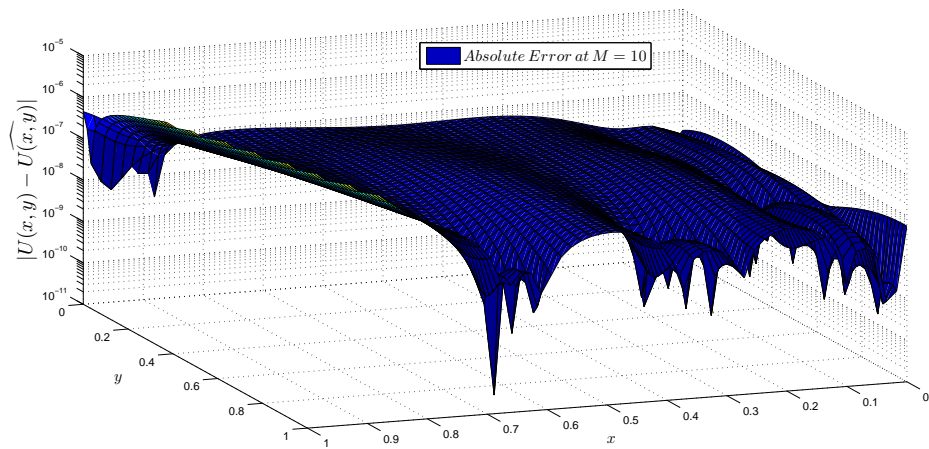Figure 5.10: Comparison is given between approximate solution and exact solution of Example 5.2.3

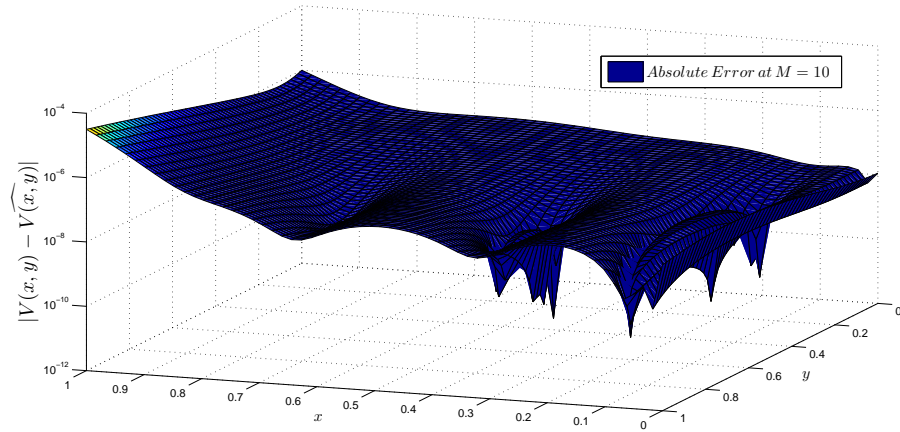Figure 5.11: At scale level $M = 10$, the error of approximation is computed for $U(x,y)$ of the Example 5.2.3.

Figure 5.12: At scale level $M = 10$, the error of approximation is computed for $V(x, y)$ of the Example 5.2.3.

High dimensional FOPDEs have numerous applications in the existing literature. Many engineering phenomenon are being modeled applying high dimensional FOPDEs, for example, the phenomena of cooling and heating of building structures, the phenomena of water reservoirs and heating lakes by radiations, the phenomena of heating of solid surfaces in materials processing, the phenomena of cyclic heating of laminated steel during pickling, and many more, see [19, 70, 83, 170].

The study of high dimensional FOPDEs is relatively very much complicated as compared to 2-dimensional FOPDEs. Many researchers have devoted their intellect in the development of the numerical schemes of high dimensional FOPDEs. Kulish [116] developed a proficient numerical method based on laplace transform for the approximate solution of high dimensional integer order PDEs. M. Akbarzade [3] applied Homotopy analysis method to develop an approximate scheme for high dimensional transient state heat conduction equation. But the developed method [3] based on very much complicated and complex algorithms and is not easy to apply on multi-terms FOPDEs. Povstenko [153] applied the integral transform method to construct the axi-symmetric solutions for a time fractional heat conduction equation corresponding to Robin BCs. Currently, Wu [182, 183] studied fractional diffusion equation and developed the numerical scheme for it with the aid of variational iteration technique. The reader can also consult [4, 144, 154] to see more results on the development of the high dimensional FOPDEs.

For application purpose, we consider the following multi-terms three dimensional heat conduction equation in a generalized form as

$$\frac{\partial^{\sigma_1} U(t,x,y)}{\partial t^{\sigma_1}} = C_1 \frac{\partial^{\beta_1} U(t,x,y)}{\partial x^{\beta_1}} + C_2 \frac{\partial^{\beta_2} U(t,x,y)}{\partial y^{\beta_2}} + C_3 \frac{\partial^{\beta_3} U(t,x,y)}{\partial x^{\frac{\beta_3}{2}} \partial y^{\frac{\beta_3}{2}}} + G_1(t,x,y), \qquad (5.3.1)$$

where C_1, C_2 and C_3 are constants, $0 < \sigma_1 \leq 1$, and $t \in [0, \Delta]$, $x \in [0, \Delta]$, $y \in [0, \Delta]$, subject to the initial condition

$$U(0,x,y) = f(x,y). \qquad (5.3.2)$$

The key idea is to transform the given problems into a system of easily solvable algebraic equations that can be easily handle by any computational software. We use the software Matlab for simulation purpose. We start the development of the scheme with the brief introduction of three dimensional shifted JPs. The notion of two dimensional shifted JPs has been discussed in Section 5.1.1.

5.3.1 Shifted JPS in three dimensions

The shifted three dimensional JPs of order N on the domain $[0, \Delta] \times [0, \Delta] \times [0, \Delta]$ can be expressed in terms of the product of function of three JPs as following

$$Q_{\Delta,abc}^{(\tau_1,\tau_2)}(t,x,y) = (Q_{\Delta,a}^{(\tau_1,\tau_2)}(t))(Q_{\Delta,b}^{(\tau_1,\tau_2)}(x))(Q_{\Delta,c}^{(\tau_1,\tau_2)}(y)), a = 0,1,2,...,n, \ b = 0,1,2,...,n, \ c = 0,1,2,...,n,$$
$$(5.3.3)$$

its orthogonality condition can be determined as

$$\int_0^\Delta \int_0^\Delta \int_0^\Delta (Q_{\Delta,a}^{(\tau_1,\tau_2)}(t))(Q_{\Delta,b}^{(\tau_1,\tau_2)}(x))(Q_{\Delta,c}^{(\tau_1,\tau_2)}(y))(Q_{\Delta,d}^{(\tau_1,\tau_2)}(t))(Q_{\Delta,e}^{(\tau_1,\tau_2)}(x))(Q_{\Delta,f}^{(\tau_1,\tau_2)}(y))R_\Delta^{(\tau_1,\tau_2)}(t)R_\Delta^{(\tau_1,\tau_2)}(x)$$
$$R_\Delta^{(\tau_1,\tau_2)}(y)dtdxdy = W_{\Delta,d}^{(\tau_1,\tau_2)}\delta_{a,d}W_{\Delta,e}^{(\tau_1,\tau_2)}\delta_{b,e}W_{\Delta,f}^{(\tau_1,\tau_2)}\delta_{c,f}.$$
$$(5.3.4)$$

5.3.2 Function approximation with three dimensional shifted JPs

In this section, we assert on the idea that any function $r(t,x,y)$ defined on $[0, \Delta] \times [0, \Delta] \times [0, \Delta]$ can be approximated in terms of the series of three-dimensional JPs as follows

$$r(t,x,y) \approx \sum_{a=0}^n \sum_{b=0}^n \sum_{c=0}^n D_{abc}(Q_{\Delta,a}^{(\tau_1,\tau_2)}(t))(Q_{\Delta,b}^{(\tau_1,\tau_2)}(x))(Q_{\Delta,c}^{(\tau_1,\tau_2)}(y)). \qquad (5.3.5)$$

The coefficients D_{abc} can be deduced using the following relation

$$D_{abc} = \frac{1}{W_{\Delta,a}^{(\tau_1,\tau_2)} W_{\Delta,b}^{(\tau_1,\tau_2)} W_{\Delta,c}^{(\tau_1,\tau_2)}} \int_0^\Delta \int_0^\Delta \int_0^\Delta r(t,x,y)(Q_{\Delta,a}^{(\tau_1,\tau_2)}(t))(Q_{\Delta,b}^{(\tau_1,\tau_2)}(x))(Q_{\Delta,c}^{(\tau_1,\tau_2)}(y))$$
$$R_\Delta^{(\tau_1,\tau_2)}(t,x,y)dtdxdy.$$

$$(5.3.6)$$

Where

$$R_\Delta^{(\tau_1,\tau_2)}(t,x,y) = R_\Delta^{(\tau_1,\tau_2)}(t)R_\Delta^{(\tau_1,\tau_2)}(x)R_\Delta^{(\tau_1,\tau_2)}(y). \qquad (5.3.7)$$

In terms of vector notation (5.3.5) can also be written as

$$r(t,x,y) \approx \Phi(t)^T G_{N \times N^2} \hat{\Phi}_{N^2 \times 1}, \qquad (5.3.8)$$

where G represents the coefficient matrix and $\hat{\Phi}$ indicates the function vector related to the variables x,y, and $\Phi(t)$ shows the one dimensional Jacobi function vector related to the variable t.

5.3.3 Error Analysis

In this section we develop an analytical relationship for the error of approximation for sufficiently smooth function $r(t,x,y) \in C([0,\Delta] \times [0,\Delta] \times [0,\Delta])$. For this assume that $\prod_{N,N,N}(t,x,y)$ the space of N terms JPs and $g_{(N,N,N)}(t,x,y)$ is its best approximation in $\prod_{(N,N,N)}(t,x,y)$. Then according to the definition of the best approximation for any polynomial $f_{(N,N,N)}(t,x,y)$ of degree $\leq N$ in variable t, x and y, we can write

$$\|r(t,x,y) - r_{(N,N,N)}(t,x,y)\|_2 \leq \|r(t,x,y) - f_{(N,N,N)}(t,x,y)\|_2. \qquad (5.3.9)$$

Then by following the same procedure as adopted in [76], we can find the bounds for the errors given below

$$|r(t,x,y) - f_{(N,N,N)}(t,x,y)| \leq \frac{1}{4}(\frac{(\Delta-0)}{N})^{N+1} max_{(t,x,y)}|\frac{\partial^{N+1}r(t,x,y)}{\partial x^{N+1}}|$$
$$+ \frac{1}{4}(\frac{(\Delta-0)}{N})^{N+1} max_{(t,x,y)}|\frac{\partial^{N+1}r(t,x,y)}{\partial y^{N+1}}| + \frac{1}{4}(\frac{(\Delta-0)}{N})^{N+1} max_{(t,x,y)}|\frac{\partial^{N+1}r(t,x,y)}{\partial t^{N+1}(N+1)!}|$$
$$- \frac{1}{64}(\frac{(\Delta-0)}{N})^{N+1}(\frac{(\Delta-0)}{N})^{N+1}(\frac{(\Delta-0)}{N})^{N+1} max_{(t,x,y)}|\frac{\partial^{3N+3}r(t,x,y)}{\partial x^{N+1}\partial y^{N+1}\partial t^{N+1}}| \qquad (5.3.10)$$
$$+ \frac{1}{16} \max_{(t,x,y)} |\frac{\partial^{2N+2}}{\partial x^{N+1}\partial y^{N+1}}r(t,x,y)|.$$

Let $\lambda_1 = max_{(t,x,y)}|\frac{\partial^{N+1}r(t,x,y)}{\partial x^{N+1}}|$, $\lambda_2 = max_{(t,x,y)}|\frac{\partial^{N+1}r(t,x,y)}{\partial y^{N+1}}|$, $\lambda_3 = max_{(t,x,y)}|\frac{\partial^{N+1}r(t,x,y)}{\partial t^{N+1}}|$, $\lambda_4 = max_{(t,x,y)}|\frac{\partial^{3N+3}r(t,x,y)}{\partial x^{N+1}\partial y^{N+1}\partial t^{N+1}}|$, and $\lambda_5 = \max_{(t,x,y)} |\frac{\partial^{2N+2}}{\partial x^{N+1}\partial y^{N+1}}r(t,x,y)|$. Then, we have the following

relation

$$|r(t,x,y) - f_{(N,N,N)}(t,x,y)| \leq \frac{1}{4}(\frac{(\Delta-0)}{N})^{N+1}\lambda_1$$
$$+ \frac{1}{4}(\frac{(\Delta-0)}{N})^{N+1}\lambda_2 + \frac{1}{4}(\frac{(\Delta-0)}{N})^{N+1}\lambda_3 \qquad (5.3.11)$$
$$- \frac{1}{64}(\frac{(\Delta-0)}{N})^{N+1}(\frac{(\Delta-0)}{N})^{N+1}(\frac{(\Delta-0)}{N})^{N+1}\lambda_4 + \frac{1}{16}(\frac{(\Delta-0)}{N})^{N+1}(\frac{(\Delta-0)}{N})^{N+1}\lambda_5.$$

Finally using (5.3.9), we have the following relation

$$\|r(t,x,y) - r_{(N,N,N)}(t,x,y)\|_2 \leq \sqrt[3]{(\Delta-0)(\Delta-0)(\Delta-0)}\Big(\frac{1}{4}(\frac{(\Delta-0)}{N})^{N+1}\lambda_1$$
$$+ \frac{1}{4}(\frac{(\Delta-0)}{N})^{N+1}\lambda_2 + \frac{1}{4}(\frac{(\Delta-0)}{N})^{N+1}\lambda_3$$
$$- \frac{1}{64}(\frac{(\Delta-0)}{N})^{N+1}(\frac{(\Delta-0)}{N})^{N+1}(\frac{(\Delta-0)}{N})^{N+1}\lambda_4 \qquad (5.3.12)$$
$$+ \frac{1}{16}(\frac{(\Delta-0)}{N})^{N+1}(\frac{(\Delta-0)}{N})^{N+1}\lambda_5\Big).$$

5.3.4 Application of the operational matrices for generalized FOPDEs

$$\frac{\partial^{\sigma_1}U(t,x,y)}{\partial t^{\sigma_1}} = C_1\frac{\partial^{\beta_1}U(t,x,y)}{\partial x^{\beta_1}} + C_2\frac{\partial^{\beta_2}U(t,x,y)}{\partial y^{\beta_2}} + C_3\frac{\partial^{\beta_3}U(t,x,y)}{\partial x^{\frac{\beta_3}{2}}\partial y^{\frac{\beta_3}{2}}} + G_1(t,x,y), \qquad (5.3.13)$$

where C_1, C_2 and C_3 are constants, $0 < \sigma_1 \leq 1$. The time and space variables are defined as $t \in [0,\Delta]$, $x \in [0,\Delta]$, $y \in [0,\Delta]$. Our main objective is the development of the numerical scheme subject to the following initial condition

$$U(0,x,y) = f(x,y). \qquad (5.3.14)$$

We seek the solution of the problem in terms of shifted JPs such that the following holds

$$\frac{\partial^{\sigma_1}U(t,x,y)}{\partial t^{\sigma_1}} \approx \hat{\Phi}(t)J_{N\times N^2}\hat{\Phi}(x,y). \qquad (5.3.15)$$

Applying fractional integration of order σ_1 with respect to t on (5.3.15), we have

$$I^{\sigma_1}\frac{\partial U(t,x,y)}{\partial t^{\sigma_1}} \approx \hat{\Phi}(t)Q_{N\times N}^{\sigma_1\Delta}J_{N\times N^2}\hat{\Phi}(x,y),$$

which implies that

$$U(t,x,y) - c_1 \approx \hat{\Phi}(t)Q_{M\times M}^{\sigma_1\Delta}K_{M\times M^2}\hat{\Phi}(x,y). \qquad (5.3.16)$$

The initial condition $U(0,x,y) = f(x,y)$ yields $c_1 = f(x,y)$ and the above equation can be written as

$$U(t,x,y) \approx \hat{\Phi}(t)Q_{N\times N}^{\sigma_1\Delta}J_{N\times N^2}\hat{\Phi}(x,y) + f(x,y), \qquad (5.3.17)$$

which implies that

$$U(t,x,y) \approx \hat{\Phi}(t)(Q^{\sigma_1 \Delta}_{N \times N} J_{N \times N^2} + F_{N \times N^2})\hat{\Phi}(x,y). \qquad (5.3.18)$$

Using the above estimate of $U(t,x,y)$, we get

$$\frac{\partial^{\beta_1} U(t,x,y)}{\partial x_1^{\beta}} \approx \hat{\Phi}(t)(Q^{\sigma_1 \Delta}_{N \times N} J_{N \times N^2} + F_{N \times N^2})G^{(\beta_1,x)}\hat{\Phi}(x,y), \qquad (5.3.19)$$

and

$$\frac{\partial^{\beta_2} U(t,x,y)}{\partial y_2^{\beta}} \approx \hat{\Phi}(t)(Q^{\sigma_1 \Delta}_{N \times N} J_{N \times N^2} + F_{N \times N^2})G^{(\beta_2,y)}\hat{\Phi}(x,y). \qquad (5.3.20)$$

$$\frac{\partial^{\beta_3} U(t,x,y)}{\partial x^{\frac{\beta_3}{2}} \partial y^{\frac{\beta_3}{2}}} \approx \hat{\Phi}(t)(Q^{\sigma_1 \Delta}_{N \times N} J_{N \times N^2} + F_{N \times N^2})G^{(\beta_3,x,y)}\hat{\Phi}(x,y). \qquad (5.3.21)$$

and

$$G_1(t,x,y) = \hat{\Phi}(t)\hat{G}_{N \times N^2}\hat{\Phi}(x,y) \qquad (5.3.22)$$

Using (5.3.15), (5.3.19), (5.3.20) in (5.3.13) we get

$$\hat{\Phi}(t)J_{N \times N^2}\hat{\Phi}(x,y) = C_1\hat{\Phi}(t)(Q^{\sigma_1 \Delta}_{N \times N} J_{N \times N^2} + F_{N \times N^2})G^{(\beta_1,x)}\hat{\Phi}(x,y)$$

$$+ C_2\hat{\Phi}(t)(Q^{\sigma_1 \Delta}_{N \times N} J_{N \times N^2} + F_{N \times N^2})G^{(\beta_2,y)}\hat{\Phi}(x,y)$$

$$+ C_3\hat{\Phi}(t)(Q^{\sigma_1 \Delta}_{N \times N} J_{N \times N^2} + F_{N \times N^2})G^{(\beta_3,x,y)}\hat{\Phi}(x,y) + \hat{\Phi}(t)\hat{G}_{N \times N^2}\hat{\Phi}(x,y), \qquad (5.3.23)$$

which can be rewritten as

$$\hat{\Phi}(t)(C_1(Q^{\sigma_1 \Delta}_{N \times N} J_{N \times N^2} + F_{N \times N^2})G^{(\beta_1,x)} + C_2(Q^{\sigma_1 \Delta}_{N \times N} J_{N \times N^2} + F_{N \times N^2})G^{(\beta_2,y)} +$$

$$C_3(Q^{\sigma_1 \Delta}_{N \times N} J_{N \times N^2} + F_{N \times N^2})G^{(\beta_3,x,y)} - J_{N \times N^2} + \hat{G}_{N \times N^2})\hat{\Phi}(x,y) = 0.$$

Hence it follows that

$$C_1(Q^{\sigma_1 \Delta}_{N \times N} J_{N \times N^2} + F_{N \times N^2})G^{(\beta_1,x)} + C_2(Q^{\sigma_1 \Delta}_{N \times N} J_{N \times N^2} + F_{N \times N^2})G^{(\beta_2,y)} +$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.3.24)$$

$$C_3(Q^{\sigma_1 \Delta}_{N \times N} J_{N \times N^2} + F_{N \times N^2})G^{(\beta_3,y)} - J_{N \times N^2} + \hat{G}_{N \times N^2} = 0,$$

Which can be written as

$$\mathbf{A}J_{N \times N^2}\mathbf{B} - J_{N \times N^2} = \mathbf{C} \qquad (5.3.25)$$

Which is lyapanove type matrix equation. where

$$\mathbf{A} = Q^{\sigma_1 \Delta}_{N \times N},$$

$$\mathbf{B} = C_1 G^{(\beta_1, x)} + C_2 G^{(\beta_2, y)} + C_3 G^{(\beta_3, x, y)},$$

and

$$\mathbf{C} = F_{N \times N^2} \mathbf{B} + \hat{G}_{N \times N^2},$$

which is an algebraic equation of Sylvester type and can be easily solved for the unknown matrix $J_{N \times N^2}$ using the value of J in (5.3.18) we can get the approximate solution of the problem.

5.3.5   Numerical test of the proposed scheme

In this section, some test problems are considered to examine the validity of the proposed method. Note that all the simulations are carried out using 5 Ghz processor. All the results are displayed using plots. The algorithm is designed in such a way that it can be easily simulated using any computational software. We use $MatLab$ for calculation and simulations.

Example 5.3.1. *As a first test case consider the following multi-terms FOPDEs with mixed partial derivative terms*

$$\frac{\partial^{\sigma_1} U(t, x, y)}{\partial t^{\sigma_1}} = C_1 \frac{\partial^{\beta_1} U(t, x, y)}{\partial x^{\beta_1}} + C_2 \frac{\partial^{\beta_2} U(t, x, y)}{\partial y^{\beta_2}} + C_3 \frac{\partial^{\beta_3} U(t, x, y)}{\partial x^{\frac{\beta_3}{2}} \partial y^{\frac{\beta_3}{2}}} + G_1(t, x, y), \qquad (5.3.26)$$

corresponding to the following initial condition

$$U(0, x, y) = x^3 \, y^3, \qquad (5.3.27)$$

where $0 < \sigma_1 \leq 1$, $t \in [0, 1]$, $x \in [0, 1]$, and $y \in [0, 1]$. The source term G_1 is defined as under

$$G_1(t, x, y) = 3\, t^2 - 6\, x^3 \, y - 9\, x^2 \, y^2 - 6\, x \, y^3 \qquad (5.3.28)$$

Example 5.3.2. *As a second test case consider the following multi-terms FOPDEs with mixed partial derivative terms*

$$\frac{\partial^{\sigma_1} U(t, x, y)}{\partial t^{\sigma_1}} = C_1 \frac{\partial^{\beta_1} U(t, x, y)}{\partial x^{\beta_1}} + C_2 \frac{\partial^{\beta_2} U(t, x, y)}{\partial y^{\beta_2}} + C_3 \frac{\partial^{\beta_3} U(t, x, y)}{\partial x^{\frac{\beta_3}{2}} \partial y^{\frac{\beta_3}{2}}} + G_1(t, x, y), \qquad (5.3.29)$$

corresponding to the following initial condition

$$U(0, x, y) = \mathrm{e}^{2\, x + 3\, y}, \qquad (5.3.30)$$

where $0 < \sigma_1 \leq 1$, $t \in [0, 1]$, $x \in [0, 1]$, and $y \in [0, 1]$. The source term G_1 is considered to be zero here.

5.3.6 Results and Discussions

The exact solution of the first test Example 5.3.1 for $\sigma_1 = 1$ is given by $t^3 + x^3 + y^3$. We compare the exact and approximate solution of the Example 5.3.1 at different values of t by taking scale level $M = 5$ applying our new method. We observe that our approximate solution is in a good agreement with the exact solution of Example 5.3.1 at $t = 0.1, t = 0.4, t = 0.7$, and $t = 1$. The results are shown in Figure 5.17. We also check the behaviour of the approximate solution of Example 5.3.1 by fixing $(x, y) = (0.1, 0.1)$ and at some fractional values of σ_1. We observe that as $\sigma_1 \to 1$, the approximate solution of Example 5.3.1 approaches to the exact solution. The results are displayed in Figure 5.18. We also do the error analysis of Example 5.3.1 at different values of t, x and y. We note that even at very low scale level $M = 5$, the absolute error between exact and approximate solution of Example 5.3.1 is very negligible. The results are plotted in Figures 5.19-5.21.

The exact solution of Example 5.3.2 is $e^{2x+3y+4t}$. Using our new numerical scheme, we compare the exact and approximate solution of Example 5.3.2 at different values of t. It is to be noted that the exact and approximate solution is in a good agreement at $t = 0.1, t = 0.4$, and $t = 1$. The outcomes are displayed in Figure 5.22. At some fractional values of σ_1, the exact and approximate solution of Example 5.3.2 is compared by fixing $(x, y) = (0.5, 0.5)$. It is evident that our approximate solution approaches to the exact solution as $\sigma_1 \to 1$. The results are shown in Figure 5.23. The error analysis of the Example 5.3.2 is also done at different scale level and at different values of t. It is observed that as the scale level increases the absolute error between exact and approximate solution of Example 5.3.2 decreases in a great extent. The results are displayed in Figures 5.24-5.25.
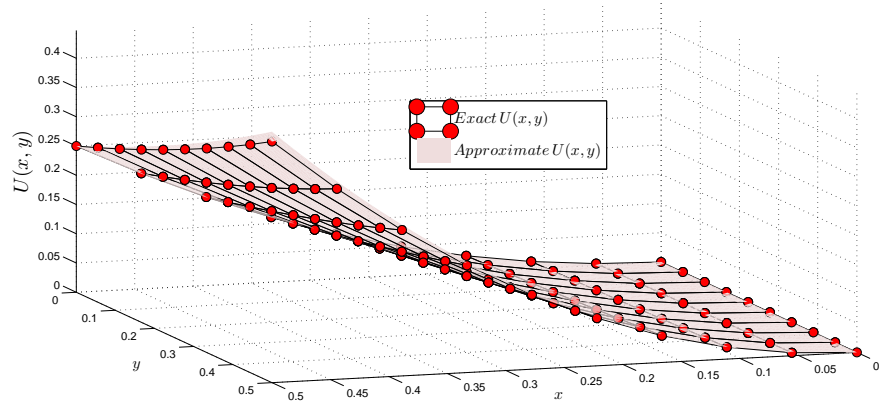
Figure 5.13: Comparison is given between approximate solution and exact solution of Example 5.2.4
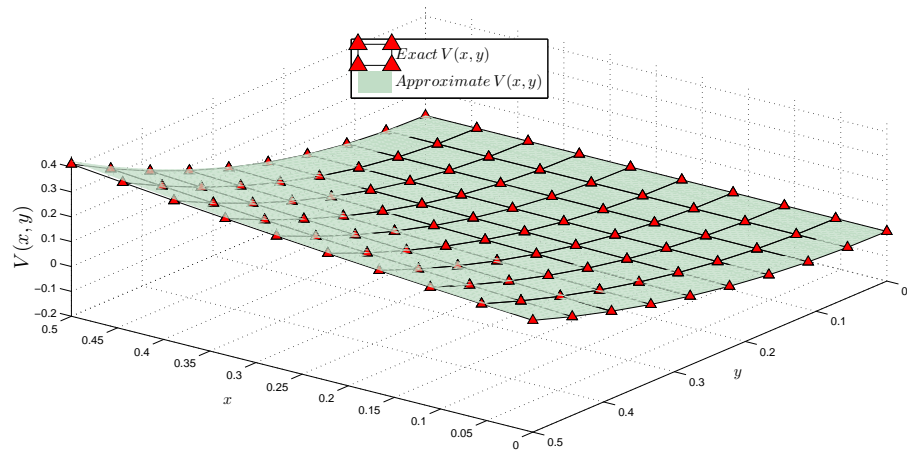

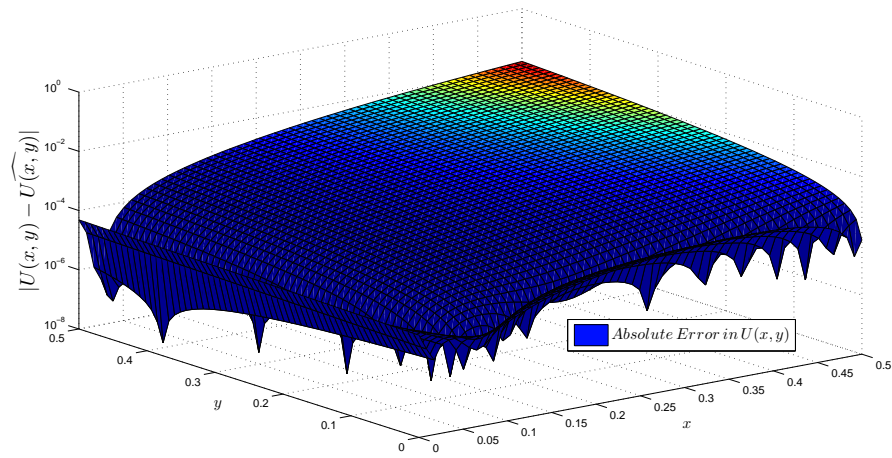Figure 5.14: Comparison is given between approximate solution and exact solution of Example 5.2.4

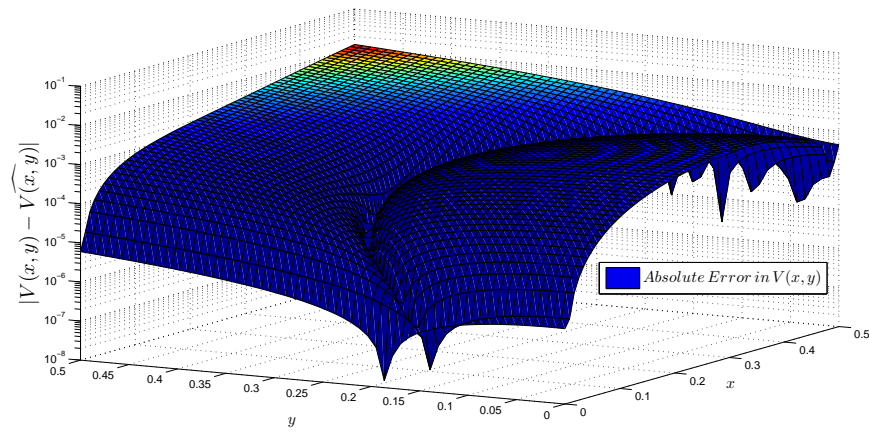Figure 5.15: Error of approximation in $U(x, y)$ of Example 5.2.4

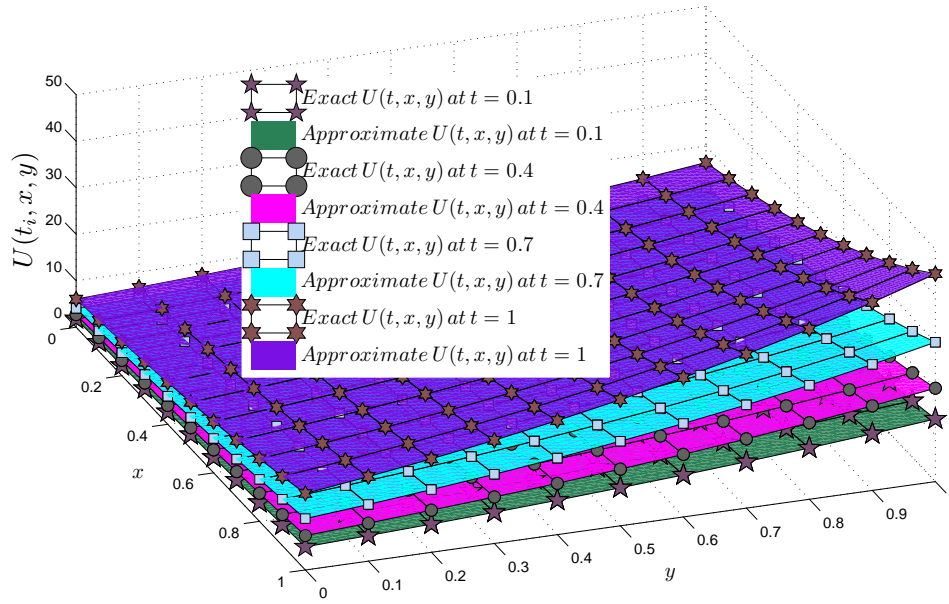Figure 5.16: Error of approximation in $V(x, y)$ of Example 5.2.4

Figure 5.17: At multiple values of t, approximate solution and exact solution of Example 5.3.1 are compared by choosing scale level $M = 5$, and JPs parameters $\tau_1 = 2 = \tau_2$.

Figure 5.18: For various values of σ_1 and by fixing $(x, y) = (0.3, 0.3)$, comparison of approximate solution with exact solution is presented for Example 5.3.1.
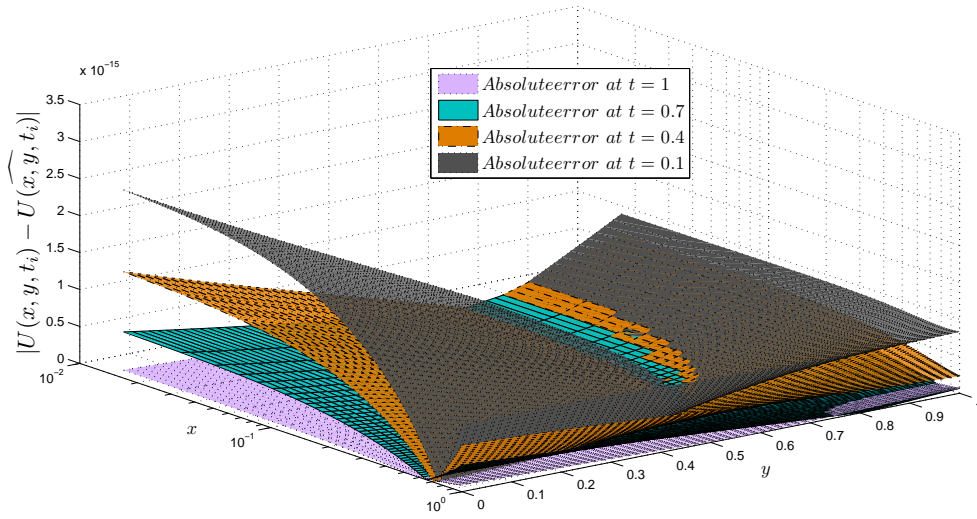
Figure 5.19: Choosing scale level $M = 5$, and for various values of t, the absolute error for Example 5.3.1 is computed.
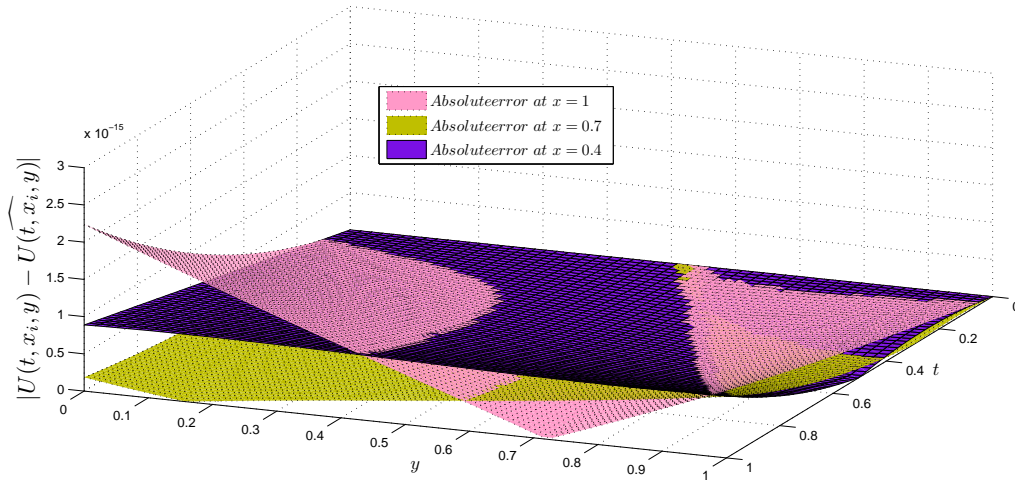
Figure 5.20: Choosing scale level $M = 5$, and for various values of x, the absolute error for Example 5.3.1 is computed.
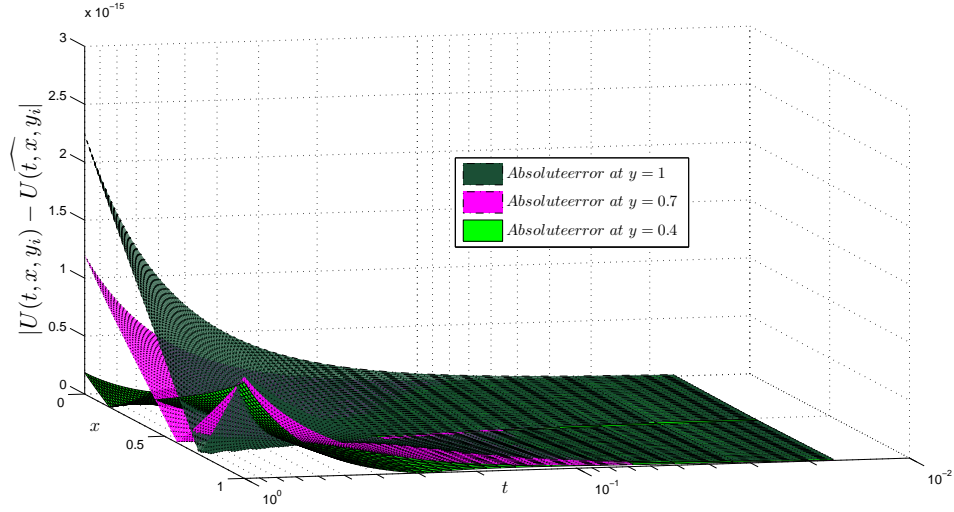
Figure 5.21: Choosing scale level $M = 5$, and for various values of y, the absolute error for Example 5.3.1 is computed.
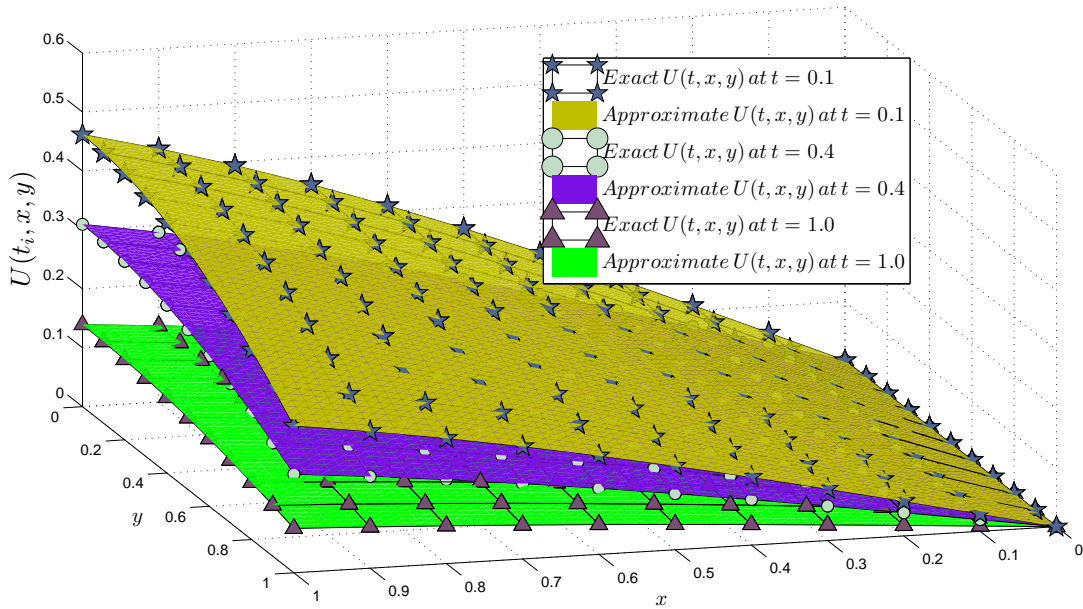
Figure 5.22: At multiple values of t, approximate solution and exact solution of Example 5.3.2 are compared by choosing scale level $M = 9$.
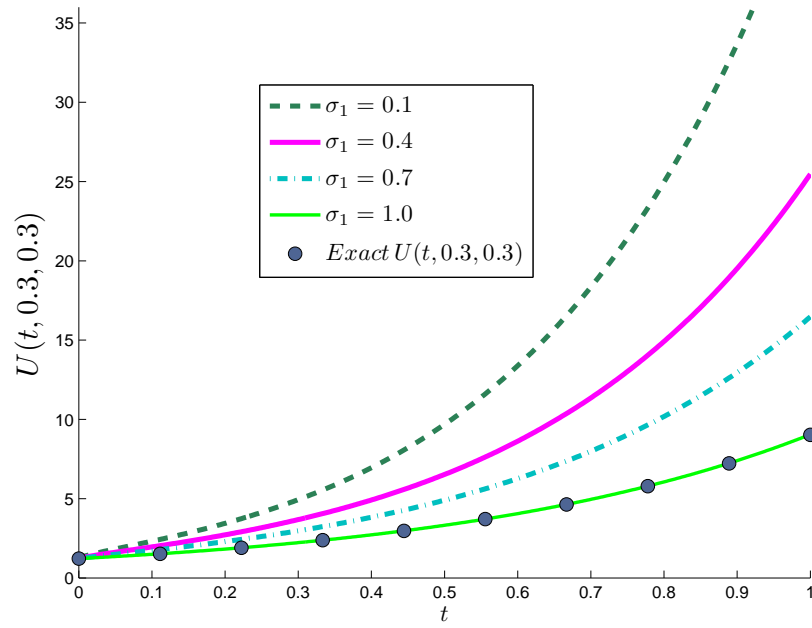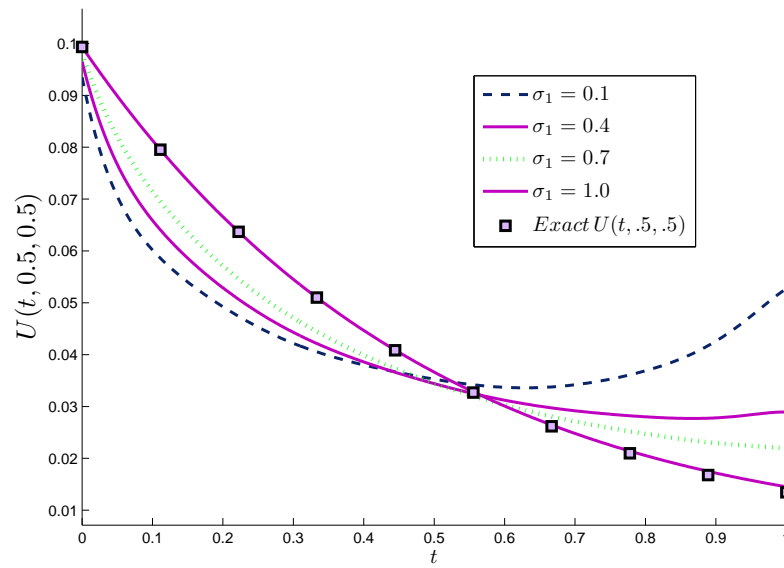
Figure 5.23: For various values of σ_1 and by fixing $(x, y) = (0.5, 0.5)$, comparison of approximate solution with exact solution is presented for Example 5.3.2.
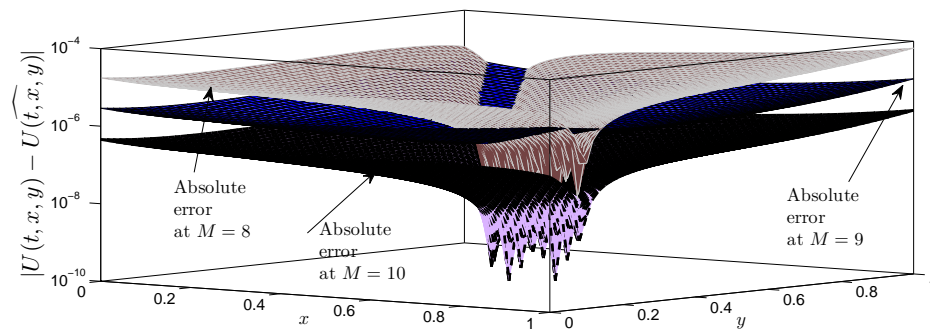

Figure 5.24: At various scala level M, and at $t = 0.5$, absolute error is computed for Example 5.3.2.
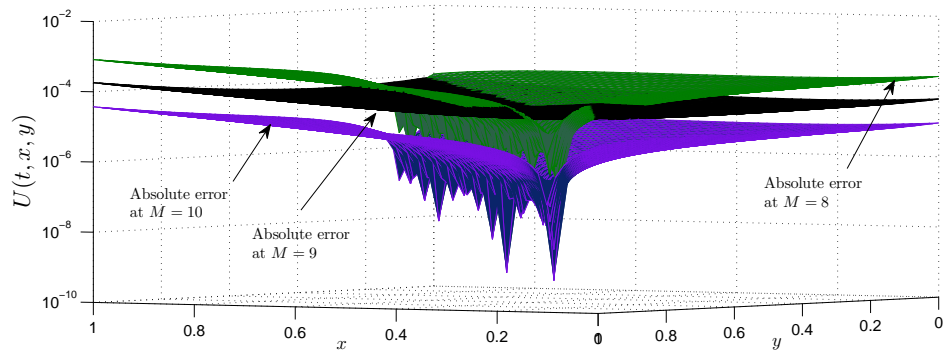
Figure 5.25: At various scala level M, and at $t = 0.8$, absolute error is computed for Example 5.3.2.

Chapter 6

Summary and Conclusion

This thesis comprises of two parts. The first part addresses the existence results for the nonlinear ordinary coupled BVPs subject to nonlinear CBCs using LUSs approach. This part unifies the treatment of the existence results of many important linear and nonlinear BVPs such as periodic, anti-periodic, Dirichlet, and Neumann by applying some monotonicity assumptions on the arguments of the certain type of BCs . The second part provides the efficient and reliable numerical methods to find the approximate solutions of the multi-terms FOPDEs and their coupled systems. In this part, the two-parametric orthogonal shifted JPs are used to establish the new operational matrices of fractional order integrals and derivatives. By means of the operational matrices, the considered FOPDEs are reduced to an algebraic ones. Being easily solvable, the associated algebraic systems lead to finding the solutions of the considered FOPDEs. This part of the thesis also deals with the development of the numerical schemes for the high dimensional multi-terms FOPDEs. The theoretical results included in the both part of the thesis are verified by taking many test examples.

In Chapter 1, a comprehensive introduction on coupled LUSs approach is discussed along-with a short historical note on the subject fractional calculus. Some concepts form approximation theory and matrix theory are also studied in this chapter. Some classical results from Fixed Point Theory are included that are much useful to establish the existence of solutions of the considered nonlinear BVPs. Some properties of LUSs are also discussed.

In Chapter 2, with the aid of Schauder's fixed point theorem, we ensure the existence of at least one fixed point of a continuous and compact map, $L^{-1}N : C^1[0,1] \times C^1[0,1] \rightarrow C^1[0,1] \times C^1[0,1]$. Under the approach of LUSs, we ensure that the fixed point of the map $L^{-1}N$, which is a solution of

the first-order nonlinear coupled BVPs lies in a closed set defined by well ordered lower and upper solutions. We unify the treatment of the existence results of very important BVPs, for example periodic and anti-periodic by imposing some monotonicity assumptions on the arguments of the nonlinear boundary functions and using the idea of coupled LUSs. We observe that our developed results are very much generalized because they satisfy the classical existence criterion of many linear and nonlinear BVPs, like periodic and anti-periodic BVPs. The existence results are also studied when LUSs are in the reverse order. Some examples are studied to ensure the validity of our developed theoretical results.

In Chapter 3, we extend the results of the Chapter 2 for the second-order nonlinear ordinary coupled BVPs with nonlinear CBCs. The existence of at least one fixed point of the continuous and compact map, $L^{-1}N : C^2[0,1] \times C^2[0,1] \to C^2[0,1] \times C^2[0,1]$ is ensured by Schauder's fixed point theorem, and the compactness of the map, $L^{-1}N$ is claimed under the outcomes of Arzelà-Ascoli theorem. We present the general approach of existence of solutions of many important linear and nonlinear BVPs, including periodic, anti-periodic, Dirichlet and Neumann in a unified way, which avoids treating these problems on a case-by-case basis. Several examples are treated to ensure the validity of the developed theoretical results.

In Chapter 4, we extend the results of Chapter 3 for the nonlinear ordinary second-order coupled BVPs with dependence of the nonlinear functions on the first-order derivative. Under the approach of LUSs method, the existence of at least one solution is ensured in the closed set defined by well ordered LUSs. We also ensure that the nonlinear functions having derivative dependence satisfy the Nagumo condition relative to the closed intervals defined by ordered LUSs. Some examples are studied to ensure the validity of the developed theoretical results.

It is very much evident that exact solutions of every fractional order problem are not always possible to determine. Therefore, there is a need to develop the reliable and efficient numerical methods that exhibit the very close agreement of the approximate solutions with the exact solutions of the considered fractional order problems. In this thesis we develop the efficient and reliable numerical methods based on operational matrices of fractional order integrals and derivatives to find the approximate solutions of multi-terms FOPDEs. The handsome amount of the test examples are taken to ensure the accuracy and efficiency of the proposed methods at different scale level. It is

to be noted that with the increase in scale level the absolute error between exact and approximate solution is decreased in a considerable extent. Also we check the behaviour of the proposed method at different fractional values of the order of the concerned FODPs and it is very much clear that as the order of the concerned problem approaches to the integer order, the approximate solution approaches to the exact solution with zero error. All the results have been shown in the form of Tables and Plots. The numerical analysis of the above discussion is elaborated in Chapter 5.

Appendix

Matlab Codes For Simulation of the Operational Matrices

Algorithm to compute function vector:

Function vector $\hat{\Phi}_{M,\Delta}^{(\tau_1,\tau_2)}$ of orthogonal shifted JPs with parameters τ_1 and τ_2 is generated by using the following algorithm:

```
function K=funvec(t,M,tau1,tau2,Delta);
syms   l tau1 tau2 t Delta;
for  j=0:M-1;
    a1=(-1)^(j-k);
    a2=gamma(j+tau2+1);
    a3=gamma(j+k+tau1+tau2+1);
d1=gamma(k+tau2+1);
d2=gamma(j+tau1+tau2+1);
d3=gamma(j-k+1);
d4=gamma(k+1);
d5=Delta^k;
BB=(a1*a2*a3)/(d1*d2*d3*d4*d5);

    K(j+1)= symsum( BB*t^k,k,0,j);
end
K=transpose(K)
```

Where "M" is the size of the matrix and known as scale level of the numerical scheme. The parameters "tau1, tau2" indicates the Jacobian parameters (τ_1 and τ_2), and "Delta" indicates the right end pint of the domain set $[0, \Delta]$.

Algorithm to generate weight function:

The weight function $R_\Delta^{(\tau_1,\tau_2)}$ is generated by the use of the following code:

```
function R=weight(t,tau1,tau2,Delta)
```

```
R=t^tau2*(Delta-t)^tau1
```

Where the parameters "tau1, tau2" indicates the Jacobian parameters (τ_1 and τ_2), and "Delta" indicates the right end pint of the domain set $[0, \Delta]$.

Algorithm to generate orthogonality constant:

The orthogonality constant $W_{\Delta,j}^{(\tau_1,\tau_2)}$ is generated by the use of the following algorithm:

```
function D=ort(i,tau1,tau2,Delta);
a1=Delta^(tau1+tau2+1);
a2=gamma(j+tau1+1);
a3=gamma(j+tau2+1);
d1=(2*j+tau1+tau2+1);
d2=gamma(j+1);
d3=gamma(j+tau1+tau2+1);
c=(a1*a2*a3)/(d1*d2*d3)
```

Where the parameters "tau1, tau2" indicates the Jacobian parameters (τ_1 and τ_2), and "Delta" indicates the right end pint of the domain set $[0, \Delta]$.

Algorithm to generate coefficient vector:

The coefficient vector D_b can be generated using the following algorithm:

```
function F=coefvec(f,t,M,tau1,tau2,Delta)
syms t
R= weight(t,tau1,tau2,Delta);
K=funvec(t,M,tau1,tau2,Delta);
Kw=f*K*R;
for j=0:M-1;
    D=ort(j,tau1,tau2,Delta);
    aa=KR(j+1);

    F(j+1)=(1/D)*int( aa ,t,0,Delta);
end
```

Where the parameters "tau1, tau2" indicates the Jacobian parameters (τ_1 and τ_2), and "Delta" indicates the right end pint of the domain set $[0, \Delta]$.

Algorithm to generate operational matrix of fractional order integral:

The operational matrix of fractional integral $\hat{Q}_{M^2 \times M^2}^{\rho,\Delta}$ of order ρ is generated with the aid of the following algorithm:

```
function Q=jacobinta(rho,M,tau1,tau2,Delta)
syms p q
for c=0:M-1;
    for e=0:M-1;
        gg=((-1)^(c-q))*gamma(c+tau2+1)*gamma(c+q+tau1+tau2+1);
        qq= gamma(q+tau2+1)*gamma(c+tau1+tau2+1)*gamma(c-q+1)*gamma(q+rho+1);
        n1n1=((-1)^(e-p))*gamma(e+p+tau1+tau2+1)*gamma(tau1+1);
        n2n2=gamma(p+rho+tau2+1)*(2*e+tau1+tau2+1)*gamma(e+1)*(Delta^rho);
        nn=n1n1*n2n2;
        m1m1=gamma(e+tau1+1)*gamma(p+tau2+1)*gamma(e-p+1)*gamma(p+1);
        m2m2=*gamma(q+p+tau1+tau2+rho+2);
        mm=m1m1*m2m2;
        Q(c+1,e+1)=symsum(((gg/qq)*symsum((cc/mm),p,0,e)),q,0,c);
    end
end
```

Where "M" is the size of the matrix and known as scale level of the numerical scheme. The parameters "tau1, tau2" indicates the Jacobian parameters (τ_1 and τ_2), and "Delta" indicates the right end pint of the domain set $[0, \Delta]$. Moreover ρ represents the order of the integral.

Algorithm to generate operational matrix of fractional order derivative:

The operational matrix of fractional derivative $\hat{D}_{M^2 \times M^2}^{\sigma,\Delta}$ of order σ is generated by using the following algorithm:

```
function D=jacobder(sigma,M,tau1,tau2,Delta)
syms p q
bba=ceil(sigma);
for c=0:M-1;
    for e=0:M-1;
        if c<bba;
            D(j+1,i+1)=0;
        else;
        aa=((-1)^(j-l))*gamma(j+tau2+1)*gamma(j+l+tau1+tau2+1);
```

```
bb= gamma( l+tau2+1)*gamma( j+tau1+tau2+1)*gamma( j−l+1)*gamma( l−sigma+1);
c1c1=((−1)^( i−k ))*gamma( i+k+tau1+tau2+1)*gamma( a+1);
c2c2=(2*i+tau1+tau2+1)*gamma( i+1)*(Delta^(−sigma ));
c3c3=gamma( k+l−sigma+tau2+1);
cc=c1c1*c2c2*c3c3 ;
d1d1=gamma( i+tau1+1)*gamma( k+tau2+1)*gamma( i−k+1);
d2d2=gamma( k+1)*gamma( l+k+tau1+tau2−sigma+2);
dd=d1d1*d2d2 ;
D( j+1, i+1)=symsum((( aa/bb )*symsum(( cc /dd ) , k ,0 , i )) , l , bba , j );


end
end
end
```

Where "M" is the size of the matrix and known as scale level of the numerical scheme. The parameters "tau1, tau2" indicates the Jacobian parameters (τ_1 and τ_2), and "Delta" indicates the right end pint of the domain set $[0, \Delta]$. Moreover, σ indicates the order of the derivative.

The codes for two and three dimensional function vectors along-with the rest of the other codes used in this thesis are left as an exercise for the reader.

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